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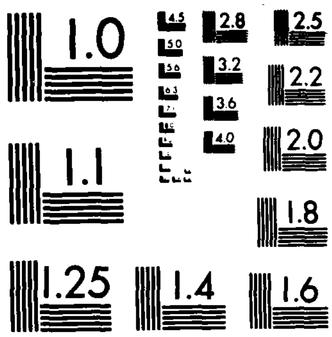

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THESIS

FORMULATION AND ANALYSIS OF SOME COMBAT-LOGISTICS PROBLEMS

by

Abdul-Latif Rashid Al-Zayani

September 1986

Thesis Advisor: Donald P. Gaver

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Models are developed to study the readiness and subsequent combat performance of an air-interceptor squadron facing sudden attack. These models necessarily link combat with logistics. The models are mainly analytical and not a Monte Carlo simulation, and can be used to indicate the optimal weapon system to be procured and to study the effect of peace-time decisions on combat outcomes. The logistics models use the matrix-geometric approach to study the general multivariate repairman problem, with the possibility of simultaneous component failures. A repairman assignment problem is formulated and solved using a multivariate continuous-time Markov decision process. Surprise scenarios are analyzed and represented explicitly. Air-to-air combat is modelled as a transient multivariate continuous-time discrete-state Markov process. Diffusion theory is used to approximate the solutions. The reason for using

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diffusions is ease of interpretation and computational economy. A comparison with simulation results shows that diffusion yields good approximations. Improvement to the diffusion approximation is provided by applying "large deviations" procedures.

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Formulation and Analysis of Some Combat-Logistics Problems

by

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ABSTRACT

Models are developed to study the readiness and subsequent combat performance of an air-interceptor squadron facing sudden attack. These models necessarily link combat with logistics. The models are mainly analytical and not a Monte Carlo simulation, and can be used to indicate the optimal weapon system to be procured and to study the effect of peacetime decisions on combat outcomes. The logistics models use the matrix-geometric approach to study the general multivariate repairman problem, with the possibility of simultaneous component failures. A repairman assignment problem is formulated and solved using a multivariate continuous-time Markov decision process. Surprise scenarios are analyzed and represented explicitly. Air-to-air combat is modelled as a transient multivariate continuous-time discrete-state Markov process. Diffusion theory is used to approximate the solutions. The reason for using diffusions is ease of interpretation and computational economy. A comparison with simulation results shows that diffusion yields good approximations. Improvement to the diffusion approximation is provided by applying "large deviations" procedures.

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I. INTRODUCTION

A. GENERAL

Defense decision makers face a difficult planning and management task: assuring that their peacetime decisions result in maximum effectiveness of their combat units in a wartime environment. It is apparent that the defense decisions made today have great impact on a nation's security in the future.

Since the end of World War II, the primary emphasis of logistics modelling has been to minimize cost under peacetime conditions. This is a consequence of a long period of relatively universal peacetime conditions; the enormous cost of modern sophisticated weapon systems; the long lead times, from the procurement to actual deployment, of these systems; and the general reduction in defense budgets over the years (with the exception of the most recent historical period). As a result, the logistics component of military resource requirements is becoming increasingly costly, with control loosely related to operational performance. Furthermore, it is highly vulnerable to military budget cutting (Drezner and Hillestad, 1982).

Defense decisions must be objectively oriented. Uninstructed and unaided intuition alone is insufficient. Models must be a part of the decision-making process.

Drezner and Hillestad (1982) point out that, as a result of economic emphasis, and lack of distinct relationships to the operational performance, models are constructed around peacetime efficiency, by considering systems and policies that are based on least-cost peacetime alternatives. Rich, Stanley and Anderson (1984) point out that logistics planning must now more directly and explicitly consider wartime threats and activities. In their words:

> The weapon system development community should develop analytical approaches and tools to evaluate trade-offs in operating concepts founded on alternative basing and support arrangements. Moreover, a weapon system's support characteristics should receive increased emphasis and clearer articulation in the formal expression of system requirements in the design and development stage. That emphasis must be matched by a more concerted use of logistics capability assessment models during concept formulation and advanced development and by earlier testing and evaluation aimed at uncovering the correcting weapon system support characteristics that could limit the system's combat effectiveness.

We endorse the above views, and believe that the models constructed and analyzed in this thesis provide a step in the indicated direction.

The main objective of this thesis is to develop analytical models that will assist in making decisions regarding the design or procurement and operation of a defensive weapon system.

This thesis analyzes an air defense weapon system, in particular an air-interceptor squadron, and constructs combat-logistics models for its performance. In general

these models or their extensions can be applied to other types of weapon systems and warfare scenarios.

B. BACKGROUND

We start by describing the characteristics of the real problem facing the defender, sketching the main features we choose to model, and stating the decision problems we address in this thesis.

The real problem as faced by the defender consists of the following components:

(a) Threat: There exists the possibility that the enemy will carry out a hostile air attack against a small country (or region) with a single defending air base.

(b) Air Defense System: In order to defend the country from a hostile air attack, the defender needs to design (or procure) and operate an air defense system. The air defense system is to consist of an air-interceptor squadron, short-medium range surface-to-air missiles (SAM), an early warning system, communication command and control (C^3) system, and an intelligence system.

(c) Early Warning System: The early warning system consists of a long-range ground-based radar with identification friendly or foe (IFF) capability. The system perhaps uses information obtained from an airborne warning and control system (AWACS) operated by friendly forces.

(d) Intelligence System: Intelligence information can be provided by monitoring enemy movements and deployments, monitoring enemy military training activities and their military communications. Analysis of the information obtained provides expectations about enemy intentions.

(e) Logistics Systems: The logistics system consists of repair shops and a supply depot. Each repair shop is specialized in repairing a certain type of module, e.g., engine, radar, airframe, etc. Each shop is assigned a certain number of "repairmen." We use the term "repairmen" to stand for repair units; the

limitations on the units' capacity for work may actually be instrumentation, tools, or the availability of spare parts. The supply depot is provided on base.

The air interceptor squadron is assigned peacetime missions, such as training and reconnaissance. The equipment on the interceptor is susceptible to failure during such (essential) operations. Thus some of the air-craft assigned to the squadron may not be available at a given moment.

The logistics system is responsible for maintaining maximum aircraft readiness; the aircrew, the C^3, and the early warning systems assist in causing maximum destruction to enemy bombers. Clearly, there is a need for models that are capable of answering questions concerning the effectiveness of new weapon systems in combat, and of analyzing the tradeoffs between the various components of the weapon system. The combat-logistics models presented here contribute to fulfilling this need.

This research constructs and analyzes some combat-logistics models for the air-interceptor squadrons. The combat-logistics models integrate logistics models with combat models.

The objectives of combat-logistics modelling are:

(i) to determine the optimal weapon system: given a specific threat, what is the optimal weapon system to be procured in order to optimize a certain measure of effectiveness (MOE) in combat? This question is to be answered with consideration paid to budget, manpower and storage space limitations.

(ii) to provide an approach for evaluating the effects of changes in the logistics system (in non-combat periods) on the effectiveness of the weapon system (in combat periods).

C. APPROACH

The approach chosen for developing the combat-logistics model is:

1. To analyze the aircraft squadron logistics system and construct analytical models to represent it.
2. To model and analyze the effect of considering different surprise scenarios.
3. To develop analytical air-to-air combat models that can represent the type of combat the squadron is expected to undergo in case of enemy attack.
4. To integrate the models constructed in (1) and (2) with the combat models constructed in (3) above, and use mathematical programming techniques to optimize the effectiveness of the squadron subject to anticipated "real life" conditions and cost constraints.

D. LITERATURE REVIEW

Our interest is mainly in the analytical side of the general problem outlined, and we review some of the work that has been done on related problems.

Multi-echelon inventory problems have recently received considerable attention by logistics analysts. This is because the multi-echelon problem captures many logistics factors such as: the effect of component shortages on weapon system availability, the stockage levels at the supply depot, repair capability, the existence of appropriate and reliable test equipment, manpower, and transportation. Zangwill (1966) developed a deterministic model to

determine the optimal production schedule for a multi-product, multi-facility production and inventory problem. Sherbrooke (1968) took a major step in analyzing multi-echelon supply systems by developing the Multi-Echelon Technique for Recoverable Items Control (METRIC), which is an analytical model of a base-depot supply system. The METRIC model assumes that the item demand is a stationary compound Poisson process with a mean value estimated by a Bayesian procedure. A failed item is repaired on base with probability β, or sent for depot repair with probability $1-\beta$. If an item is sent for depot repair, the base generates a resupply request on the depot. The model assumes no lateral resupply between bases. The model minimizes the expected backorders for any system investment, and computes the optimal redistribution of stock.

Muckstadt (1973) extended the results of METRIC to incorporate a multi-indenture (subcomponents of components) relationship. Mod-METRIC computes both base and depot spare stock levels for all items and all indenture levels.

A further extension of METRIC, which considers non-stationary demand processes, is the Dyna-METRIC model, see Hillestad (1982). For a description of Dyna-METRIC from a logistician's perspective, the reader is referred to Pyles (1984). The Dyna-METRIC model assesses the contribution of the components' support processes to wartime aircraft availability and sortie rate.

Drezner and Hillestad (1982) point out that a major weakness of work in the logistics area for military applications is the lack of consideration of the weapon system availability and capability. The use of backorder measures for determining stockage levels does not consider the importance of the various components of the weapon systems. Backorder measures are common in the METRIC family of models.

We now mention several features of the maintenance activity that can be incorporated into our logistics models.

One widely used method of increasing operational availability is to cannibalize weapon systems, substituting good components from one downed aircraft to prevent another aircraft from becoming unavailable. However, this area has received little attention in the literature, although the situation of full instantaneous cannibalization was considered (see Fisher and Brennan (1986)). A pioneer work for modelling cannibalization is by Hirsch, Meisner and Boll (1968). The most recent work in this area is by Fisher and Brennan (1986), where several cannibalization policies are compared using simulation. Our study includes consideration of cannibalization strategies.

Logistics models are evolving toward the wartime support aspects of logistics but more relevant measures of military performance must be considered. Rich (1986) points out that we cannot be satisfied with such measures as aircraft

availability and sortie generation capability. Instead, we need to focus on measures of system output such as targets killed. The observation has helped motivate the research reported in this thesis.

A problem that currently faces defense decision makers is the extensive specialization, and the resulting size, of the maintenance force. An F-16 wing, for example, includes 1089 support personnel organized into 23 different specialties; see Rich, Stanley and Anderson (1984). This situation leads to vulnerability of the forces being supported and stresses manpower resources. An alternative that decision makers usually consider is <u>cross training</u> of the maintenance personnel so that they become generalists, i.e., capable of repairing more than one type of system (e.g., propulsion system, hydraulic system, fuel system, etc.). This alternative becomes more attractive as the technology evolves, and systems are designed to be more modular. Increasingly, repairs at the base level consist of fault detection and replacement of the faulty modules. The faulty modules are then sent to the depot for actual repair.

Rich, Stanley and Anderson (1984) also point out that combat forces must be able to operate in forward areas with minimal support resources. This means streamlining the flightline workforce by training and using maintenance generalists.

If the above alternative is employed, then one of the major questions that will face the decision makers is: how to manage the maintenance force so that the air-interceptor squadron readiness is sustained at the maximum possible level? Chapter III of this thesis addresses this question by searching for an optimal repairmen assignment policy that maximizes the expected number of aircraft ready to engage in combat. We do this by formulating the problem as a multivariate-continuous-time Markovian decision process.

Recently, Szarkowicz and Knowles (1985) considered the problem of determining optimal operating policies for an M/M/S queueing system. The system state is represented by a bivariate vector, the first component is the number of customers in the system, and the second component is the number of active servers. They used a continuous time Markov decision process formulation to obtain an optimal control policy for the number of active servers; the cost structure includes customer holding and server operating costs as well as a linear switching cost.

The problem that Sparkowicz and Knowles analyzed differs from our problem, since we consider more than one type of item to be repaired, and we need to have an item of each type to be operational in order to have an operational aircraft.

For more work related to this area, the reader is referred to Howard (1969), Zuckerman (1986), Jo (1984), Crabill (1972) and Miller (1968a), (1968b) and (1969).

To consider the effect of the logistics system and the allocation of fixed monetary resources to competing but interrelated demands (such as, spares, test equipment, personnel, etc.) on the combat outcome of the weapon system, we need to consider the combat process explicitly in the model.

In order to keep the model analytically attractive, we chose to represent the logistics system at a lower level of detail than the METRIC family of models, and to introduce the combat model explicitly. The logistics model is therefore relatively simple. It presents only the most dominant features of a combat-logistics process (i.e., the process that the weapon system undergoes in peace and wartime environments). This research effort may be regarded as a prototype of a large family of more complete combat-logistics models. Taylor (1980) points out that a simple model may yield an understanding of important relations that are difficult to perceive in a more complex model, and such insights provide valuable guidance for higher-resolution computerized investigations.

Since the combat model plays a role of the same importance as the logistics model, an entire chapter of this thesis is devoted to the combat model. In particular, our

approach is to provide efficient computational methods to obtain "exact" solutions to multivariate Markov models, supplementing these with approximate, easily comprehended approximations based on diffusions.

Recently, Taylor (1983) presented a comprehensive treatise on Lanchester-type models for combat, i.e., differential-equation models of attrition in force-on-force combat operations in which both deterministic and stochastic models for combat are considered. Taylor (1983) states, "combat is anything but a deterministic process." Thus, in order to incorporate the random nature of combat into the model, the combat process must be represented by a stochastic process. Combat analysts tend to represent the combat process probabilistically by Markov processes; see for example, Taylor (1983), Kimble (1970), Clark (1969), and Bhat (1984).

The most recent work done in this area is by Karmeshu and Taiswal (1986), where they incorporate the effects of environmental fluctuations by regarding the parameters of the problem as stochastic processes.

Feigin, Pinkas, and Shinar (1984) proposed a simple Markov model for the analysis of many-on-many air combat. The model is a four variate-Markov process where the state variables are: the number of free blue planes, the number of free red planes, the number of pursuing blue planes, and the number of pursuing red planes. Their solution approach

was to construct and solve the forward Chapman-Kolmogorov equations.

We choose to model the air-to-air combat process by generalizing the deterministic model so that some probability statements can be evaluated. This can be done by representing the combat process by a multivariate diffusion process. We find that the diffusion-combat models constructed in this thesis provide good approximations to the corresponding Markovian-combat model. For the definition, description and discussion of diffusion processes the reader is referred to Karlin and Taylor (1981). For related work the reader should refer to the papers of Iglehart (1965), McNeil (1973), Schach and McNeil (1973), Gaver and Lehoczky (1975), (1977a), (1977b), (1977c), (1979), and Gaver and Jacobs (1985).

E. ORGANIZATION OF THE THESIS

This thesis models a problem that is complex and has many factors to be considered. Therefore, we choose to decompose the problem into components: a logistics problem, and a combat problem; we then analyze each one separately, build an analytical model for each, then integrate (or link) the various models to evaluate the system effectiveness, which is, in combat terms, some measure of "leakage" or survivorship of the attackers. Each chapter analyzes and develops a model for a specific subproblem. Chapter II models the general repair system. Chapter III studies and

models the repairman assignment problem, while Chapter IV studies and models the air-to-air combat problem. Chapter V provides the approach for linking the logistics model to the combat model. Figure (1.1) illustrates the general structure of the thesis.

In the first part of Chapter II, we build a model of a simple logistics system. We assume that each aircraft requires a single module in order to be considered operational. The problem is modelled as a birth-and-death process, and also as a closed migration process. A closed form solution for the limiting distribution is presented. This is a simplified version of the real problem; it serves as a starting point or feasibility study.

The surprise phenomenon is then analyzed and classified as total or partial surprise. Depending on what type of surprise is considered, the squadron readiness probability distribution is then computed. A simple example is provided to illustrate the model.

In the second part of Chapter II, we generalize the above model to represent a two-module logistics system; i.e., it is assumed that each aircraft requires two operational modules (one of each type) in order for it to be considered operational. We find that a simultaneous failure mechanism must be introduced in order to have a realistic representation. A simultaneous failure mechanism means that the conditional probability of an aircraft failure due to

LOGISTICS MODELS

* Operational Logistics (Chapter II)
  ** Single module logistics (1ML). Aircraft needs single module to be considered operational.
  ** Two module logistics (2ML). Aircraft requires 2 types of modules (one of each) to be considered operational.
  ** Surprise:
    * Full surprise
    * Partial surprise

* Repairmen allocation problem (Chapter III)
  ** Linear Programming
  ** Dynamic Programming

COMBAT MODELS (Chapter IV)

* Deterministic
* Diffusion
* Markovian
* Simulation
* Large Deviations

COMBAT-LOGISTICS MODELS (Chapter V)

* Single-module-combat-logistics: Integrates 1ML with a combat model.
* Two-module-combat-logistics: Integrates 2ML with a combat model.

Figure 1.1. Structure of the Thesis

both modules failing is positive. The problem is formulated as a bivariate birth-and-death process. To simplify the computations the problem is then formulated so that it can be presented in a matrix-geometric form. This is done by adopting the arguments in Gani and Purdue (1984) and in Gaver, Jacobs and Latouche (1984). For an explanation of the matrix-geometric formulation the reader is referred to Neuts (1981). The squadron readiness probability distribution for the two-module logistics system is then presented for both types of surprise. This is followed by a simple example to illustrate the technique.

In Chapter III, we hypothesize an operational logistics scenario. The problem is a repairmen allocation problem, where the repairmen are assumed to be general technicians capable of repairing both types of modules. The maintenance supervisor is interested in finding the optimal repair policy that maximizes the squadron readiness. The problem is formulated as a bivariate-continuous-time-Markov-decision process. A one-to-one transformation of a multivariate-Markov process to a univariate-Markov process is presented. The transformation allows us to use the existing theory and techniques for the univariate-Markov decision process to solve a multivariate one. The problem is then solved by dynamic programming and linear programming approaches. An example to illustrate both approaches is given, followed by

a comparison and evaluation of the results of both approaches.

In Chapter IV we discuss the characteristics of the air-to-air combat and provide the following air-to-air combat scenarios to be modelled:

(i) BCD Scenario: Both defenders and enemy bombers are assumed to be vulnerable, and defenders spend some time searching for a free enemy bomber. Once a free enemy bomber is detected, it is then engaged by a free defender. The one-to-one engagement is assumed to continue until either a bomber or a defender is killed. If the bomber is killed the defender once again searches for a free bomber; the process continues.

(ii) I Scenario: This scenario assumes that the defenders are invulnerable. Only one free defender can engage a free bomber, and the engagement is assumed to be instantaneous; i.e., once a defender becomes free, no time is spent searching for a free bomber if one is available (perfect C^3 system). This is the simplest situation; it is characterized by a one-dimensional Markov chain.

Both of the above scenarios implicitly assume that the enemy bombers arrive in combat simultaneously.

In the second part of Chapter IV, we construct and analyze deterministic models for the above two scenarios. Next we generalize the deterministic models by developing diffusion models to represent the air-to-air combat. Simple continuous-time discrete-state Markov models are then constructed to represent the combat processes for the above scenarios; the latter models are referred to as Markovian-Combat models (of course the diffusions are an approximation to these, and are themselves Markovian; the reason for using diffusions is ease of interpretation and computational

simplicity). The forward Chapman-Kolmogorov equations for these models are derived, as are their Laplace transforms. Techniques for solving the Laplace transforms are presented.

In order to evaluate and compare the different approaches, the Markovian combat process for the BCD scenario is simulated. A large deviations equation for the MOE, under the I scenario, is derived; for the large deviations approach see Feller (1971), and for applications see Mazumdar and Gaver (1984).

Finally we present in Chapter IV some illustrations to compare and analyze the combat models.

In Chapter V we study the entire combat-logistics situation that confronts the decision maker. We then present the model's basic concept and assumptions. The problem is then formulated as a non-linear-integer mathematical programming model.

We first examine the single-module combat logistics (1MCL) model which is a result of linking the single-module logistics model to a combat model, subject to budget and operational constraints. The two-module combat logistics (2MCL) model is next constructed as a generalization for 1MCL.

By way of illustration we present some examples for 1MCL and 2MCL models using both the BCD and the I scenarios. Some sensitivity analysis on the combat and logistics parameters are also conducted.

We close our discussion in Chapter VI with a review of the results obtained and their implications, and some concluding remarks and recommendations for further research.

Two appendices are included to describe additional but somewhat peripheral research related to the main emphasis of the thesis.

Appendix A contains a discussion of the large deviations technique. A large deviations equation for fixed, and randomized, sums of independent and identically distributed random variables is presented, followed by applications to compound Poisson process and renewal processes. The techniques have application to classical inventory control problems. A proposed approach to minimize the large deviations error is introduced, followed by an illustration of the approach applied to the compound Poisson process.

In Appendix B, we generalize Lanchester's linear law by developing a diffusion model. A simple continuous-time discrete-state Markov model is then constructed to represent the corresponding combat process.

II. OPERATIONAL LOGISTICS MODELS

A. INTRODUCTION

This chapter formulates and analyzes some operational logistics problems. Logistics is an area that gives rise to classical operational research problems which suggests that a considerable body of work has been done in this area. Nevertheless, the linkage between combat models and logistics is a research area that is ripe for further research. Particular attention in this thesis has been directed towards the repairman problem. Feller (1971) defines the repairman problem, and Kelley (1979) provides a closed product-form solution for the single type module repairman problem. This chapter formulates and solves a multivariate repairman problem with simultaneous failure and consequent simultaneous demand for several resources.

Gaver and Lehoscky (1976) formulated and analyzed a 2-module repairman problem, wherein there was a number, a, of aircraft, each requiring both types of modules, one of each, to operate. When an aircraft fails it has conditional probabilities p_1, p_2 and p_{12} of being down due to a failure of Type 1, Type 2, or both types of modules, respectively, where $p_1 + p_2 + p_{12} = 1$. This model assumed that an aircraft will be in repair until the failed modules were repaired (i.e., the assumption indicates that there were no spare modules assigned to the organization). The problem was modelled as a diffusion process which

gave a good approximation to long-run aircraft availability when compared with simulation model results. Their work tends to validate the use of a convenient approximate model (diffusion) for decision making.

The single-type module repairman problem is presented first in this chapter followed by the two-type module repairman problem. After discussing the objectives of this chapter, the models for the various types of problems will then be constructed followed by sample solutions. The probability distribution function for the interceptor squadron readiness will be calculated. Those results will then be used in Chapter V for the blending of the logistics and combat models.

B. OBJECTIVE

The objective of this chapter is to formulate and analyze the generalized repairman problem for the maintenance organization of an interceptor squadron. Analysis of the repairman problem means determining the various factors affecting the repair process in the organization. A major factor is the number of different types of modules requiring different types of repair facilities and different repair times. This chapter analyzes the following repairman problems:

- a single type of module.
- two types of modules with independent failure rates.
- two types of modules with dependent failure rates, with the probability of a simultaneous failure of both modules being negligible.
- two types of modules with a non-zero simultaneous failure rate.

C. SCOPE

An air interceptor squadron is selected to represent the logistical situation under study. The squadron is considered to be assigned a number, a, of aircraft. The study concentrates on finding the readiness of the aircraft at the time of attack by enemy bombers. This readiness depends upon the total number of each module type and the number and rates of repair characteristic of the repairmen in each shop. In turn, the readiness that results influences combat effectiveness.

D. APPROACH

To achieve the above objective, the logistics system of the squadron during a "peacetime" (noncombat) situation must be analyzed. The repair shop experiences demands for repair, depending on the type of problem under study. If the problem formulation considered assumes that the whole aircraft is a module, then the shop experiences a single type of demand for repair.

Let the number of modules up at time t be $\{X(t), t \geq 0\}$; under Markov conditions the variable X(t) can be considered to be the state of the system. Because of the random nature of the system state change, the process $\{X(t); t \geq 0\}$ represents a stochastic process. By introducing certain reasonable assumptions that will be stated later, the process can be modelled as a continuous-time Markov process.

However, if the problem under study assumes that an aircraft requires each of two types of modules to qualify as a mission capable aircraft, then the two-shop maintenance organization

may experience either simultaneous or independent demands for two types of repair, depending on the case under consideration. Let the number of modules of type i, i = 1,2 for repair at time t be $\{X_i(t), t \geq 0, i = 1,2\}$. Then the bivariate vector having components $X_1(t)$ and $X_2(t)$ can be considered to be the state of the system at time t. If full cannibalization is allowed, then $X_1(t)$ and $X_2(t)$ determine the number of aircraft operational. As is true for the univariate case, this process may be modelled as a bivariate-continuous-time Markov process. More generally, if we have a multi-module, e.g., K-module, setup, the process could be modelled by a K^{th}-variate-continuous-time Markovian process.

E. SINGLE-MODULE PROBLEM FORMULATION

Consider an aircraft squadron that consists of a number, a, of failure-prone aircraft. Each will be considered not mission capable if, and only if, a specific module is not working. Let λ denote the overall Markovian failure rate of an aircraft, where failure of an aircraft means a failure of the module it currently carries. Suppose that there are M modules assigned to the organization where $M \geq a$, and there are R repairmen capable of repairing the modules, and only one repairman is required to repair a failed module.

Let X(t) denote the number of modules that are installed or available at time t. Thus the number of mission capable aircraft is $\min\{a, X(t)\}$, which is the number of aircraft that are failure-prone, the others awaiting a module. Finally, it

is assumed that repair is Markovian (or exponential), where μ denotes the rate at which an individual repair is completed.

Figure (2.1) shows a schematic of the above system. Karlin and Taylor (1984) have analyzed the above problem and have shown that once $X(t)$ is known, the number of modules in any category can be determined. A module can be in any of four states:

(i) Operating (installed on aircraft).

(ii) "Up" but not operating, i.e., a spare.

(iii) Under repair.

(iv) Waiting for repair.

Each of the modules down at time t must either be undergoing or awaiting repair. Thus:

(i) The number of modules operating is $\min\{a, X(t)\}$.

(ii) The number of spares is $\max\{0, X(t)-a\}$.

(iii) The number of modules in repair is $\min\{R, M-X(t)\}$.

(iv) The number waiting for repair is $\max\{0, M-X(t)-R\}$.

The number of modules in any category can therefore be determined easily once $X(t)$ is known.

The transition probabilities of the process $\{X(t),\ t \geq 0\}$, of order dt are given below:

t		t+dt	Probability	
i	→	i+1	$\mu \min\{R, M-i\}$	
	→	i-1	$\lambda \min\{a, i\}$	(2.1)

The above assumptions and formulation specify the stochastic process $\{X(t),\ t \geq 0\}$ as a birth-and-death process with

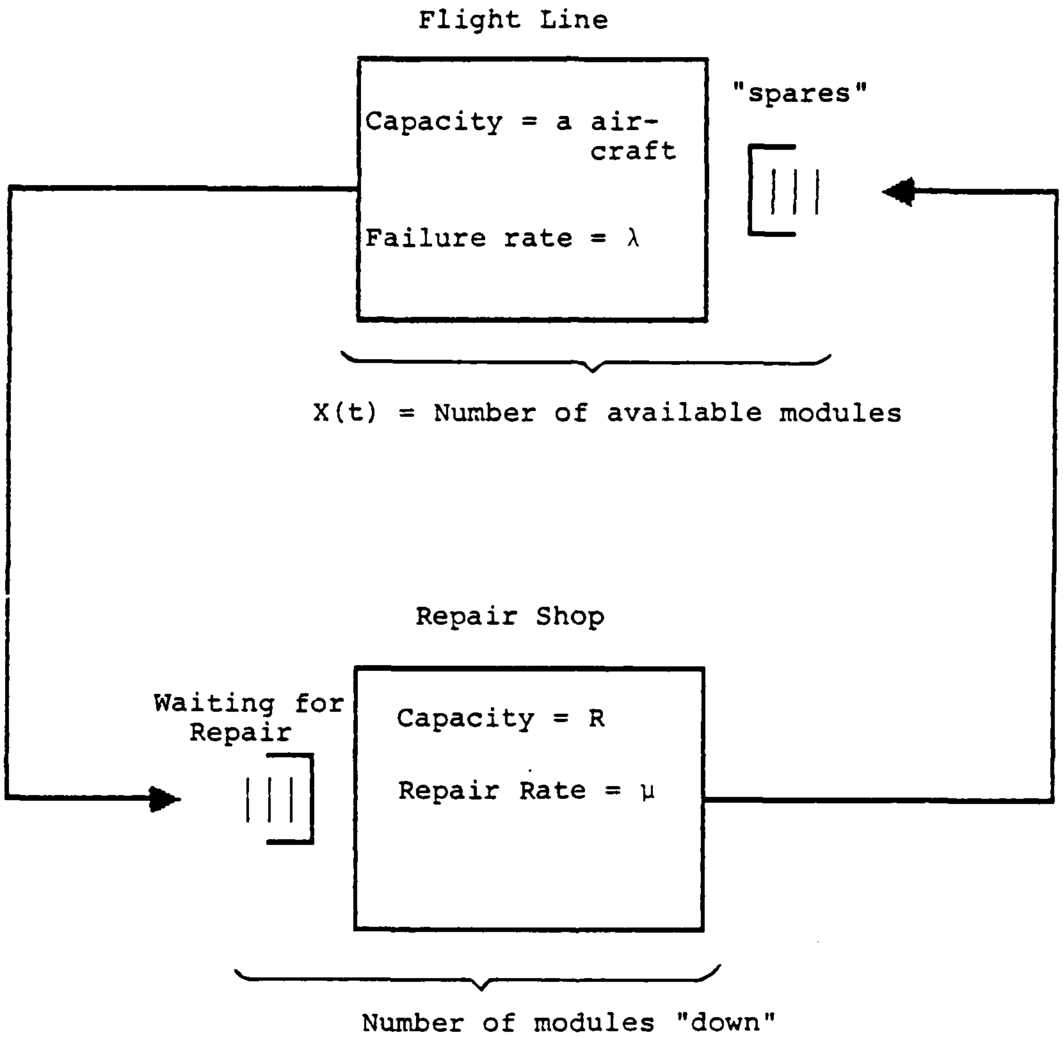

Figure 2.1. Schematic of the Single-Module Logistics System

state space $\{i = 0,1,2,\ldots,M\}$. The parameters of the process are given by:

$$\lambda_i = \lambda \times \min\{a,i\} = \begin{cases} \lambda i & \text{for } i = 0,1,2,\ldots,a-1 \\ \lambda a & \text{for } i = a,\ldots,M \end{cases}$$

and

$$\mu_i = \mu \times \min\{R,M-i\} = \begin{cases} \mu R & \text{for } i = 0,1,2,\ldots,M-R \\ \mu \times (M-i) & \text{for } i = M-R+1,\ldots,M \end{cases}$$

Let Q be the infinitesimal generator for the Markov process $\{X(t),\ t \geq 0\}$. Then Q will be given by the matrix on the next page. Define $P_{ij}(t)$ to be the conditional probability of the process being in state j given that it started in state i. Then $P_{ij}(t) = P\{X(t) = j \mid X(0) = i\}$. To simplify the notation we suppress the initial conditions and write,

$$P_j(t) = P_{ij}(t)$$

The forward Chapman-Kolmogorov equations are derived by writing the probability that the process is in state j at time d+dt as a function of the state probabilities at time t. The probability of being in state j at time t+dt is expressed by:

$$P_j(t+dt) = (1-(\lambda_j+\mu_j)dt)P_j(t) + \mu_{j-1}dtP_{j-1}(t) + \lambda_{j+1}dtP_{j+1}(t) + o(dt)$$

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ M-1 \\ M \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & 2 & & & & & M-1 & M \end{array} \\
\left[\begin{array}{ccccccccc}
-\mu_0 & \mu_0 & & & & & & & \\
\lambda_1 & -(\lambda_1+\mu_1) & \mu_1 & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & \lambda_i & -(\lambda_i+\mu_i) & \mu_i & & & \\
 & & & & & & & & \\
 & & & & & & \lambda_{M-1} & -(\lambda_{M-1}+\mu_{M-1}) & \mu_{M-1} \\
 & & & & & & & \lambda_M & -\lambda_M
\end{array}\right]
\end{array}
$$

Subtracting $P_j(t)$ from both sides and dividing by dt and letting dt tend to zero results in the following:

$$\frac{dP_j(t)}{dt} = -(\lambda_j+\mu_j)P_j(t) + \mu_{j-1}P_{j-1}(t) + \lambda_{j+1}P_{j+1}(t)$$

since $\frac{o(dt)}{dt} \to 0$ as $dt \to 0$. With the initial condition:

$$P_j(0) = \begin{cases} 1 & \text{if } j = M \\ 0 & \text{otherwise} \end{cases}$$

the forward Chapman-Kolmogorov equations are:

$$\begin{aligned} \frac{dP_M(t)}{dt} &= -\lambda_M P_M(t) + \mu_{M-1}P_{M-1}(t) \\ \frac{dP_0(t)}{dt} &= -\mu_0 P_0(t) + \lambda_1 P_1(t) \\ \frac{dP_j(t)}{dt} &= -(\lambda_j+\mu_j)P_j(t) + \mu_{j-1}P_{j-1}(t) + \lambda_{j+1}P_{j+1}(t) \\ & \qquad 1 \le j \le M-1 \end{aligned} \tag{2.2}$$

with the initial condition:

$$P_j(0) = \begin{cases} 1 & \text{if } j = M \\ 0 & \text{otherwise} \end{cases}$$

F. LIMITING DISTRIBUTION

All states of the Markov process defined by (2.1) communicate so the Markov process is irreducible. Hence the limiting distribution exists and is independent of the initial condition. Since every state is a positive recurrent state, the limiting values are all positive and form a probability distribution. This results from the following theorem (Bhat, 1984):

Theorem (2.1)

(1) If the Markov process is irreducible then the limiting distribution $\lim_{t \to \infty} P_j(t) = \pi_j$ exists and is independent of the initial conditions of the process. The limits $\{\pi_j, j \in S\}$, where S is the state space, are such that they either vanish identically (i.e., $\pi_j = 0$ for all $j \in S$) or are all positive and form a probability distribution (i.e., $\pi_j > 0$ for all $j \in S$, $\sum_{j \in S} \pi_j = 1$).

(2) The limiting distribution $\{\pi_j, j \in S\}$ of an irreducible positive recurrent Markov process is given by the unique solution of the equation $\vec{\pi}Q = 0$, and $\sum_{j \in S} \pi_j = 1$, where Q is the infinitesimal generator for the process, and $\vec{\pi} = (\pi_0, \pi_1, \ldots)$.

Let π_j denote the long-run probability of the process being in state j. Therefore:

$$\pi_j = \lim_{t \to \infty} P_j(t)$$

From the above, as $t \to \infty$, $\frac{d}{dt} P_j(t) \to 0$, therefore the forward Chapman-Kolmogorov equations (2.2) become:

$$-\lambda_M \pi_M + \mu_{M-1} \pi_{M-1} = 0$$

$$-\mu_0 \pi_0 + \lambda_1 \pi_1 = 0 \tag{2.4}$$

$$-(\lambda_j + \mu_j)\pi_j + \mu_{j-1}\pi_{j-1} + \lambda_{j+1}\pi_{j+1} = 0$$

Below are given alternative methods for calculating π_j.

<u>Approach 1</u>:

The finite-state-continuous-time Markov process $\{X(t),\ t \geq 0\}$ has been shown to have a limiting distribution.

From the set of equations (2.4), π_j can be written recursively as:

$$\pi_i = \begin{cases} \frac{R\mu}{i\lambda}\,\pi_{i-1} & \text{for } i = 1,\ldots,M-R-1 \\ \frac{(M-i)}{a\lambda}\,\pi_{i-1} & \text{for } i = M-R,\ldots,M \end{cases} \tag{2.5}$$

Therefore:

$$\pi_i = \begin{cases} \frac{1}{i!}\left(\frac{R\mu}{\lambda}\right)^i \pi_0 & \text{for } i = 1,\ldots,M-R-1 \\ \frac{R!}{(M-i)!\,a!}\left(\frac{\mu}{\lambda}\right)^i \frac{R^{M-R}}{a^{i-(M-R-1)}}\,\pi_0 & \text{for } i = M-R,\ldots,M \end{cases}$$

and $\sum_{i=0}^{M} \pi_i = 1$. Hence

$$\pi_0 + \sum_{i=1}^{M-R-1} \pi_i + \sum_{i=M-R}^{M} \pi_i = 1$$

$$\pi_0 + \sum_{i=1}^{M-R-1} \frac{1}{i!}\left(\frac{R\mu}{\lambda}\right)^i \pi_0 + \sum_{i=M-R}^{M} \frac{R!}{(M-i)!a!}\left(\frac{\mu}{\lambda}\right)^i \frac{R^{M-R}}{a^{i(M-R-1)}} \pi_0 = 1$$

Thus:

$$\pi_0 = \{1 + \sum_{i=1}^{M-R-1} \frac{1}{i!}\left(\frac{R\mu}{\lambda}\right)^i + \frac{R!}{a!}R(aR)^{M-R-1} \sum_{i=M-R}^{M} \frac{1}{(M-i)!}\left(\frac{\mu}{a\lambda}\right)^i\}^{-1} \tag{2.6}$$

Therefore, the limiting distribution can be calculated by determining π_0 first using equation (2.6), and then applying equation (2.5) to calculate the limiting probabilities recursively.

Approach 2:

Another approach to computing the π_j's is to view the system as a closed migration process. Kelly (1974) shows that a closed product form can be derived for the equilibrium distribution.

The flight line and the repair shop can be considered as stations between which the modules move. Assume that modules cannot leave or enter the system, and that the total number of modules in the system, M, is fixed. Then, X(t) is a cyclic queue, where a module departing from one station joins the other with probability 1.

Using Kelly's results for a closed migration process, we get the following expressions for the limiting distribution:

$$\pi_0 = \frac{B}{\prod_{j=0}^{M-1} \mu_j}$$

$$\pi_i = \frac{B}{\prod_{j=1}^{i} \lambda_j \prod_{j=i}^{M-1} \mu_j} \qquad 1 \leq i \leq M-1 \qquad (2.7)$$

$$\pi_M = \frac{B}{\prod_{j=1}^{M} \lambda_j}$$

where:

$$B = \left[\frac{1}{\prod_{j=0}^{M-1} \mu_j} + \sum_{i=1}^{M-1} \frac{1}{\prod_{j=1}^{i} \lambda_j \prod_{j=i}^{M-1} \mu_j} + \frac{1}{\prod_{j=1}^{M} \lambda_j} \right]^{-1}$$

Readiness:

The factors affecting the combat readiness of an interceptor squadron can be categorized as:

- Factors that are related to the logistics system.
- Factors that are related to the operational system.
- Factors that are related to the intelligence or warning system.

The logistical factors include the number of spare modules authorized to keep on base, the number of maintenance personnel assigned to the maintenance organization, the skill and level of training of maintenance personnel, together with

the repair policy implemented, the number and types of test and diagnostic equipment, and the supply of spare parts. Finally, the quality of facility management is an intangible influence.

The operational factors include the number of aircraft assigned to the squadron, the number and skill of the pilots, the effectiveness of the training programs, and the sensitivity of the early warning system.

The intelligence factors include:

(a) The procedures used in collecting information about the enemy movements including the prediction of enemy actions based on current political status.

(b) Quality of the analysis of the information collected and the confidence of the decision makers in the conclusions reached. Information could indicate the intention of the enemy, however the defender sometimes fails to anticipate surprise attacks.

The effect of surprise is reduced as the readiness increases. Surprise attack can be classified in different stages as: (Brodin, 1978)

(1) Total surprise,

(2) Partial surprise,

(3) No surprise.

In an attempt to model the surprise phenomena mathematically, the following warning scenario was considered.

1. <u>Models for Readiness: Stages of Surprise</u>

a. Total Surprise

Under the Single-Module-Logistics formulation, if a total surprise phenomenon is considered, then a readiness distribution model can be derived as follows:

Let $D(0)$ = Number of interceptors ready to engage in combat at the time of attack.

Then:

$$P\{D(0) = j\} = \begin{cases} \pi_j & \text{for } 0 \le j < a \\ \sum_{i=a}^{M} \pi_i & \text{for } j = a \\ 0 & \text{otherwise} \end{cases} \qquad (2.8)$$

This assumes that attacks occur infrequently, so a steady-state distribution prevails, and that there is essentially no warning.

b. Partial Surprise

Suppose that the intelligence data analysis indicates that the enemy might strike after W units of time (e.g., days). Assume that W is a random variable distributed exponentially with mean η^{-1}. During such a time period the squadron will undergo a mobilization and preparedness stage, and the squadron activities will change. They will eliminate training and conduct air reconnaissance and early warning missions only, and they will change (enhance) the maintenance repair capability. Such changes are assumed to cause the failure rate to decrease and the repair rate to increase.

Assume that the failure rate decreases from λ to λ' where $\lambda > \lambda'$, and that the repair rate increases from μ to μ' where $\mu < \mu'$.

Let $\tilde{X}(t)$ be the number of modules available at time t with the new rate parameters. Therefore, the $\{\tilde{X}(t), t \geq 0\}$ is a continuous-time-Markov process with the following rate parameters:

$$\lambda_i' = \lambda' \times \min\{a,i\} = \begin{cases} \lambda' i & \text{for } i = 0,1,\ldots,a-1 \\ \lambda' a & \text{for } i = a,\ldots,M \end{cases}$$

and

$$\mu_i' = \mu' \times \min\{R,M-i\} = \begin{cases} \mu' R & \text{for } i = 0,1,\ldots,M-R \\ \mu'(M-i) & \text{for } i = M-R+1,\ldots,M \end{cases}$$

The initial condition for $\{\tilde{X}(t), t \geq 0\}$ is the limiting distribution for $\{X(t), t \geq 0\}$, i.e., set of equations (2.7).

$$P\{\tilde{X}(0) = i\} = \pi_i \qquad i = 0,1,2,\ldots,M \tag{2.9}$$

Therefore, $\{\tilde{X}(t), t \geq 0\}$ is a birth-and-death process with initial conditions π_i for $i = 0,1,\ldots,M$. Let

$$\tilde{P}_{i,j}(t) = P\{\tilde{X}(t) = j \mid \tilde{X}(0) = i\}$$

The forward Chapman-Kolmogorov equations for $\{\tilde{X}(t), t \geq 0\}$ are:

$$\frac{d\tilde{P}_{i,M}(t)}{dt} = -\lambda_M'\tilde{P}_{i,M}(t) + \mu_{M-1}'P_{i,M-1}(t)$$

$$\frac{d}{dt}\tilde{P}_{i,j}(t) = -(\lambda_j'+\mu_j')\tilde{P}_{i,j}(t) + \mu_{j-1}'\tilde{P}_{i,j-1}(t)$$

$$+ \lambda_{j+1}'\tilde{P}_{i,j+1}(t) \qquad (2.10)$$

$$\frac{d}{dt}\tilde{P}_{i,o}(t) = -\mu_o'P_{i,o}(t) + \lambda_1'P_{i.}(t)$$

Let $\hat{\tilde{P}}_{i,j}(s)$ be the Laplace transform for $\tilde{P}_{i,j}(t)$:

$$\hat{\tilde{P}}_{i,j}(s) = \int_0^\infty e^{-st}\,\tilde{P}_{i,j}(t)\,dt$$

To find the readiness probability distribution at the time of attack, it is required to find the probability distribution for the number of modules available at the time of attack. Let

$$\tilde{P}_{i,j}(W) = P\{\tilde{X}(W) = j \mid \tilde{X}(0) = i, W\}$$

Now note that

$$E[\tilde{P}_{i,j}(W)] = \int_0^\infty \tilde{P}_{i,j}(\omega)\eta e^{-\eta\omega}d\omega = \eta\int_0^\infty e^{-\eta\omega}\tilde{P}_{i,j}(\omega)d\omega$$

$$E[\tilde{P}_{i,j}(W)] = \eta\,\hat{\tilde{P}}_{i,j}(\eta), \qquad (2.11)$$

so that the required readiness is given in terms of a Laplace transform.

Taking the Laplace transforms of (2.10) and integrating by parts yields:

$$-(\mu_o'+s)\hat{\tilde{P}}_{i,o}(s) + \lambda_1'\hat{\tilde{P}}_{i,1}(s) = -\tilde{P}_{i,o}(0)$$

$$-(\lambda_j'+\mu_j'+s)\hat{\tilde{P}}_{i,j}(s) + \mu_{j-1}'\hat{\tilde{P}}_{i,j-1}(s)$$

$$+ \lambda_{j+1}'\hat{\tilde{P}}_{i,j+1}(s) = -\tilde{P}_{i,j}(0) \qquad (2.12)$$

$$i \leq j \leq M-1$$

$$-(\mu_M'+s)\hat{\tilde{P}}_{i,M}(s) + \mu_{M-1}'\hat{\tilde{P}}_{i,M-1}(s) = -\tilde{P}_{i,M}(0)$$

where:

$$\tilde{P}_{i,j}(0) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

The readiness probability distribution can be calculated by solving for $P\{\tilde{X}(W) = j|W\}$, $j = 0,1,\ldots,a$, from the above system of linear equations using (2.11) together with the initial distribution for the restricted process $\{\tilde{X}(t),\ t \geq 0\}$ given by (2.9).

From the law of total probability,

$$P\{\tilde{X}(W) = j|W\} = \sum_{i=0}^{M} P\{\tilde{X}(W) = j, \tilde{X}(0) = i|W\}$$

$$= \sum_{i=0}^{M} P\{\tilde{X}(W) = j|\tilde{X}(0) = i, W\}P\{\tilde{X}(0) = i\}$$

Thus:

$$P\{\tilde{X}(W) = j|W\} = \sum_{i=0}^{M} \pi_i P\{\tilde{X}(W) = j|\tilde{X}(0) = i, W\} \qquad (2.13)$$

Taking the expectation of both sides of (2.13) with respect to W, we obtain the following:

$$E[P\{\tilde{X}(W) = j|W\}] = \sum_{i=0}^{M} \pi_i E[P\{\tilde{X}(W) = j|\tilde{X}(0) = i, W\}]$$

Using equation (2.11), we get:

$$E[P\{\tilde{X}(W) = j|W\}] = \eta \sum_{i=0}^{M} \pi_i \hat{\tilde{P}}_{i,j}(\eta) \qquad (2.14)$$

Hence, the readiness probability distribution is given by:

$$
P\{D(0) = j\} = \begin{cases} \eta \sum_{i=0}^{M} \pi_i \hat{P}_{i,j}(\eta) & 0 \le j < a \\ \eta \sum_{k=a}^{M} \sum_{i=0}^{M} \pi_i \hat{P}_{i,k}(\eta) & j = a \\ 0 & \text{otherwise} \end{cases} \tag{2.15}
$$

2. <u>An Illustration for the Single-Module Logistics Model</u>

The following basic data and assumptions were used as an example for the Single-Module Logistics model illustration:

- An aircraft requires one type of module to be considered as a mission capable aircraft.
- The squadron is assigned $a = 4$ aircraft.
- The total number of modules assigned to the squadron is $M = 6$.
- The maintenance commander is assigned 2 repairmen (crews).
- An aircraft is reported down at a rate of $\lambda = 1$ aircraft per day, during peacetime, and becomes $\lambda' = 0.5$ aircraft per day, during the warning time, if a partial surprise scenario is considered.
- Modules are repaired at a rate of $\mu = 0.5$ modules per day during peacetime, and $\mu' = 1$ module per day during the warning time if a partial surprise scenario is considered.
- The expected time for the warning period (i.e., time until the attack) is $\eta^{-1} = 6$ days.

For illustration, both classes of surprise attacks will be considered.

a. Total Surprise

The limiting distribution for the number of modules up was found to be:

State: i =	0	1	2	3	4	5	6
$\pi(i)$ =	.3676	.3676	.1838	.0613	.0153	.0038	.0005

Then using equations (2.8), we can compute the squadron readiness probability distribution. This was found to be:

Number of operational aircraft: j =	0	1	2	3	4
$P\{D(0) = j\}$ =	.3676	.3676	.1838	.0613	.0196

This means that under the total surprise scenario, the probability of the squadron having no aircraft ready to engage the incoming threat is .3676, the probability of having one aircraft operational is .3676, and the probability of having two aircraft operational is .1838, etc. Note that the median number of aircraft ready to engage is less than 1; and the expected number of aircraft ready to engage is .9387 aircraft which is also less than 1. This suggests that the squadron must prepare for combat in order to survive.

b. Partial Surprise

The Laplace transform $\hat{\tilde{P}}_{i,j}(\eta)$ was found to be:

i \ j	0	1	2	3	4	5	6
0	.6748	.9240	1.1146	1.1209	.9404	.8383	.3869
1	.2310	1.0011	1.2075	1.2143	1.0188	.9082	.4192
2	.1393	.6037	1.3313	1.3389	1.1233	1.0013	.4622
3	.1051	.4554	1.0042	1.5127	1.2692	1.1314	.5222
4	.0882	.3821	.8425	1.2692	1.4843	1.3232	.6107
5	.0786	.3406	.7510	1.1314	1.3232	1.6252	.7501
6	.0725	.3144	.6932	1.0443	1.2214	1.5002	1.1539

Therefore the $E[\tilde{P}_{i,j}(W)]$ is:

i \ j	0	1	2	3	4	5	6
0	.1125	.1540	.1858	.1868	.1567	.1397	.0645
1	.0385	.1669	.2013	.2024	.1698	.1513	.0698
2	.0232	.1006	.2219	.2232	.1872	.1669	.0770
3	.0175	.0759	.1675	.2521	.2115	.1885	.0870
4	.0147	.0637	.1404	.2115	.2474	.2205	.1018
5	.0131	.0568	.1252	.1886	.2205	.2708	.1250
6	.0121	.0524	.1155	.1741	.2036	.2500	.1923

Hence,

$$P\{\tilde{X}(W) = j, \tilde{X}(0) = i \mid W\} = P\{\tilde{X}(W) = j \mid \tilde{X}(0) = i, W\} P\{\tilde{X}(0) = i\}$$

$$= \tilde{P}_{i,j}(W)\pi_i$$

Thus $P\{\tilde{X}(W) = j, \tilde{X}(0) = i | W\}$ is given by

i \ j	0	1	2	3	4	5	6
0	.0414	.0566	.0683	.0686	.0576	.0514	.0237
1	.0142	.0614	.0740	.0743	.0624	.0556	.0257
2	.0043	.0185	.0407	.0410	.0344	.0307	.0142
3	.0011	.0047	.0103	.0154	.0130	.0115	.0053
4	.0002	.0010	.0021	.0032	.0038	.0034	.0016
5	0	.0002	.0006	.0007	.0008	.0010	.0006
6	0	0	.0001	.0001	.0001	.0001	.0001

Therefore,

$$P\{\tilde{X}(W) = j | W\} = \sum_{i=0}^{6} P\{\tilde{X}(W) = j, \tilde{X}(0) = i | W\}$$

which is given by the following table:

State: j =	0	1	2	3	4	5	6
$P\{\tilde{X}(W) = j \| W\}$ =	.0612	.1424	.1961	.2033	.1721	.1537	.0712

The readiness probability distribution becomes:

Number of aircraft: j =	0	1	2	3	4
$P\{D(0) = j\}$	.0612	.1424	.1961	.2033	.3970

Note that the median number of aircraft ready to engage has improved to approximately 3; and the expected number of aircraft ready to engage has improved to 2.7325. This is a

result of incorporating the warning time, and considering the new parameters for the process.

Under both classes of the surprise scenarios, we were able to compute the readiness of the squadron measured in terms of the initial distribution for the number of aircraft ready to engage in combat.

G. TWO-MODULES PROBLEM FORMULATION

Consider an aircraft squadron that consists of a number, a, of failure-prone aircraft. Each aircraft requires each of two modules (e.g., an engine system, and an avionics system) to be considered mission capable. Let p_1, p_2 and p_{12} be the conditional probabilities that, when an aircraft is reported down, it requires respectively, Type 1, Type 2, or both types of repair, where $p_1 + p_2 + p_{12} = 1$. Let λ denote the overall Markovian failure rate of an aircraft. Suppose that there are M_1 and M_2 modules of Type 1 and Type 2 respectively assigned to the organization, where $\min\{M_1,M_2\} \geq a$. Furthermore there are R_1 and R_2 repairmen (crews) capable of repairing modules of Type 1 and Type 2, respectively, and only one repairman (crew) can work on a failed module at any time.

Let $X_1(t)$ and $X_2(t)$ denote the number of modules of Type 1 and Type 2 that are awaiting or undergoing repair at time t. Thus the number of mission-capable aircraft is $\min\{a,M_1-X_1(t), M_2-X_2(t)\}$. This is the number of aircraft that are failure prone; the others are awaiting either a module of Type 1 and/or a module of Type 2. This formulation tacitly assumes

instantaneous cannibalization, so that one aircraft will not be awaiting a Type-1 module only, while another aircraft awaiting a Type-2 module only. A Type-1 module can be cannibalized from the latter aircraft and installed on the former aircraft resulting in one aircraft operational.

Finally, it is assumed that repair in both shops is Markovian (or exponential) where μ_1 and μ_2 denote the repair rates at which individual repairs of Type-1 and Type-2 modules, respectively, are completed. Figure (2.2) shows a schematic of this system.

Once $X_1(t)$ and $X_2(t)$ are known, then the number of modules of both types in any category can be determined from the Markovian assumptions.

(i) The number of modules operating is $\min\{a, M_1 - X_1(t), M_2 - X_2(t)\}$.

(ii) The number of spares of Type i is $\max\{0, M_i - X_i(t) - a\}$, $i = 1,2$

(iii) The number of modules in repair at shop i is $\min\{R_i, X_i(t)\}$, $i = 1,2$

(iv) The number of modules of Type i waiting for repair is $\max\{0, X_i(t) - R_i\}$, $i = 1,2$

Thus the number of modules in any category can be determined once $X_i(t)$, $i = 1,2$, are known.

The transition probabilities of the process $\{X_1(t), X_2(t); t \geq 0\}$, of order dt are given below:

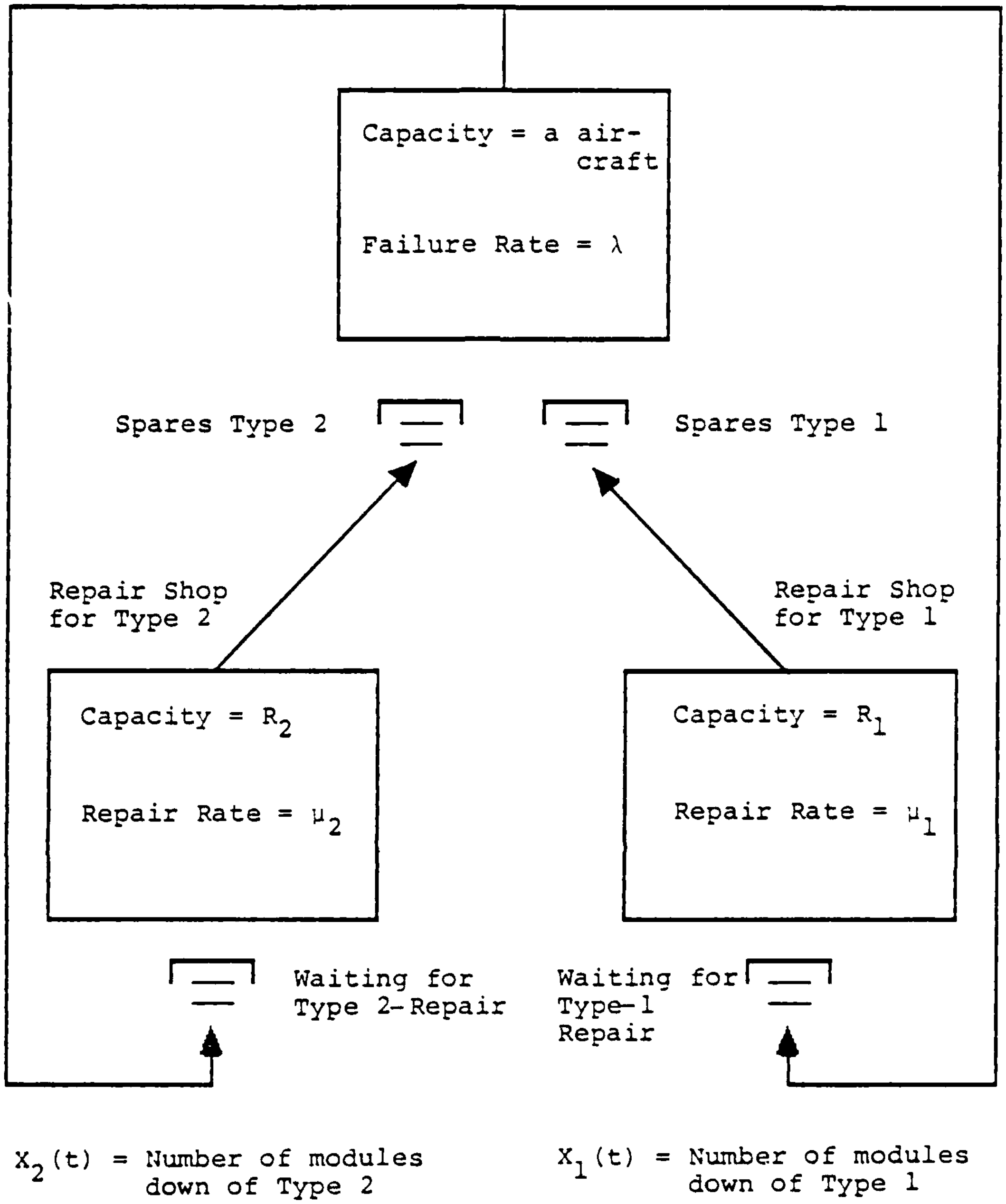

Figure 2.2. Schematic of the Two-Modules Logistics System

t		$t+dt$	Probability
(i,j)	$\to$	$(i+1,j)$	$p_1\lambda \min\{a,M_1-i,M_2-j\}dt$
	$\to$	$(i,j+1)$	$p_2\lambda \min\{a,M_1-i,M_2-j\}dt$
	$\to$	$(i+1,j+1)$	$p_{12}\lambda \min\{a,M_1-i,M_2-j\}dt$ (2.16)
	$\to$	$(i-1,j)$	$\mu_1 \min\{R_1,i\}dt$
	$\to$	$(i,j-1)$	$\mu_2 \min\{R_2,j\}dt$

$\{X_1(t),X_2(t);\ t \geq 0\}$ is a finite state space continuous-time-Markov process. All states of the process, defined by (2.16), communicate, so the Markov process is irreducible. Hence, by Theorem (2.1), the limiting distribution exists, and can be found by solving a system of linear equations. The parameters of the process are given by:

$$f(i,j) = \lambda \min\{a,M_1-i,M_2-j\}$$

$$g_1(i,j) = \begin{cases} \mu_1 i & i = 1,2,\ldots,R_1-1 \\ \mu_1 R_1 & i = R_1,\ldots,M_1 \end{cases} \tag{2.17}$$

$$g_2(i,j) = \begin{cases} \mu_2 j & j = 1,2,\ldots,R_2-1 \\ \mu_2 R_2 & j = R_2,\ldots,M_2 \end{cases}$$

To achieve the objectives of this chapter, three different formulations for the Two-Modules Logistics problems will be presented.

1. Formulation I

Under this formulation, it is assumed that the two modules fail independently. The assumption states that an aircraft requires only one of the two types to be considered a mission capable aircraft. That is, as long as there is one module, regardless of its type, installed on the aircraft then the aircraft can perform its mission. Suppose that the rate parameters are given by:

$$f_k(j) = p_k \lambda \min\{a, M_k - j\} \qquad k = 1,2$$

$$g_k(j) = \begin{cases} \mu_k j & j = 1,2,\ldots,R_k - 1 \\ \mu_k R_k & j = R_k,\ldots,M_k \end{cases} \qquad k = 1,2$$

where $p_1 + p_2 = 1$.

Under the above assumption, the marginal processes $\{X_1(t),\ t \geq 0\}$ and $\{X_2(t),\ t \geq 0\}$ are two independent birth and death processes.

Define $P^k_{ij}(t)$ to be the conditional probability of the k^{th} process, $k = 1,2$, being in state j given that it started in state i. Then

$$P^k_{ij}(t) = P\{X_k(t) = j \mid X_k(0) = i\} \qquad k = 1,2$$

Let

$$P^k_j(t) = P^k_{ij}(t)$$

The forward Chapman-Kolmogorov equations, using the same probabilistic argument as in the Single-Module Logistics formulation, are given by:

$$\frac{dP_j^k(t)}{dt} = -(f_k(j)+g_k(j))P_j^k(t) + f_k(j-1)P_{j-1}^k(t) + g_k(j+1)P_{j+1}^k(t) \qquad 1 \le j \le M_k-1$$

$$\frac{dP_0^k(t)}{dt} = -f_k(0)P_0^k(t) + g_k(1)P_1^k(t) \tag{2.18}$$

$$\frac{dP_{M_k}^k(t)}{dt} = -g_k(M_k)P_{M_k}^k(t) + f_k(M_k-1)P_{M_k-1}^k(t)$$

with initial condition:

$$P_{i,j}^k(0) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \qquad k = 1,2$$

a. Limiting Distribution for Formulation I

Let π_j^k denote the long run probability of the k^{th} process, k = 1,2, being in state j. Then:

$$\pi_j^k = \lim_{t\to\infty} P_{i,j}^k(t) \qquad k = 1,2$$

If the limits exist, then $\lim_{t\to\infty} \frac{dP_j^k(t)}{dt} = 0$, therefore, the forward Chapman-Kolmogorov equations become:

$$-f_k(0)\pi_0^k + g_k(1)\pi_1^k = 0$$

$$-(f_k(j)+g_k(j))\pi_j^k + f_k(j-1)\pi_{j-1}^k + g_k(j+1)\pi_{j+1}^k = 0$$

$$1 \le j \le M_k-1$$

$$(2.19)$$

$$-g_k(M_k)\pi_{M_k}^k + f_k(M_k-1)\pi_{M_k-1}^k = 0$$

$$\sum_{j=0}^{M_k} \pi_j^k = 1$$

where the last expression is the normalizing equation.

The marginal processes $\{X_k(t), t \ge 0\}$, $k = 1,2$, can be viewed as independent closed migration processes. Hence, the joint distribution is equal to the product of the marginals, which are given by (2.19).

Let π_{ij} be the limiting distribution for the joint process $\{X_1(t),X_2(t); t \ge 0\}$. Then:

$$\pi_{ij} = \pi_i^1\pi_j^2$$

Therefore, we get:

$$\pi_{00} = \frac{B}{\prod_{n=0}^{M_1-1} f_1(n) \prod_{n=0}^{M_2-1} f_2(n)}$$

$$\pi_{0j} = \frac{B}{\prod_{n=0}^{M_1-1} f_1(n) \prod_{n=1}^{j} g_2(n) \prod_{n=j}^{M_2-1} f_2(n)} \quad 1 \le j \le M_2-1$$

$$\pi_{i0} = \frac{B}{\prod_{n=1}^{i} g_1(n) \prod_{n=i}^{M_1-1} f_1(n) \prod_{n=0}^{M_2-1} f_2(n)} \qquad 1 \le i \le M_1-1$$

$$\pi_{ij} = \frac{B}{\prod_{n=1}^{i} g_1(n) \prod_{n=i}^{M_1-1} f_1(n) \prod_{n=1}^{j} g_2(n) \prod_{n=j}^{M_2-1} f_2(n)}$$

$$1 \le i \le M_1-1; \quad 1 \le j \le M_2-1$$

$$\pi_{0M_2} = \frac{B}{\prod_{n=0}^{M_1-1} f_1(n) \prod_{n=1}^{M_2} g_2(n)} \tag{2.20}$$

$$\pi_{M_1 0} = \frac{B}{\prod_{n=1}^{M_1} g_1(n) \prod_{n=0}^{M_2-1} f_2(n)}$$

$$\pi_{iM_2} = \frac{B}{\prod_{n=1}^{i} g_1(n) \prod_{n=i}^{M_1-1} f_1(n) \prod_{n=1}^{M_2} g_2(n)} \qquad 1 \le i \le M_1-1$$

$$\pi_{M_1 j} = \frac{B}{\prod_{n=1}^{M_1} g_1(n) \prod_{n=1}^{j} g_2(n) \prod_{n=j}^{M_2-1} f_2(n)} \qquad 1 \le j \le M_2-1$$

$$\pi_{M_1 M_2} = \frac{B}{\prod_{n=1}^{M_1} g_1(n) \prod_{n=1}^{M_2} g_2(n)}$$

where:

$$B = \prod_{k=1}^{2} \left\{ \frac{1}{\prod_{j=0}^{M_k-1} f_k(j)} + \sum_{i=1}^{M_k-1} \frac{1}{\prod_{j=1}^{i} g_k(j) \prod_{j\,i}^{M_k-1} f_k(j)} + \frac{1}{\prod_{j=1}^{M_k} g_k(j)} \right\}^{-1}$$

The assumption made under formulation I results in a very simple and unrealistic system, because an aircraft requires only one type of module to be considered as a mission capable aircraft. This formulation is impractical for aircraft squadron scenarios where the objective of the maintenance commander is to maintain the highest possible readiness in the squadron. The above formulation could be used for modelling a depot repair system where there are multiple types of modules that could be assumed to demand repair independently. A generalization of the above method that can accommodate the depot repair scenario is trivial and a closed product form solution for the limiting distribution of the joint probability can be found. Suppose there are N different types of modules. Then:

$$\pi_{n_1,\ldots,n_N} = \prod_{k=1}^{N} \frac{B}{\prod_{j=1}^{n_k} g_k(j) \prod_{j=n_k}^{M_k-1} f_k(j)}$$

where:

$$B = \prod_{k=1}^{N} \left\{ \frac{1}{\prod_{j=0}^{M_k-1} f_k(j)} + \sum_{i=1}^{M_k-1} \frac{1}{\prod_{j=1}^{i} g_k(j) \prod_{j=i}^{M_k-1} f_k(j)} + \frac{1}{\prod_{j=1}^{M_k} g_k(j)} \right\}^{-1}$$

2. Formulation II

Under this formulation, it is assumed that $p_{12} = 0$, i.e., there is no simultaneous failure of both items. Aircraft can be reported down either because of a failure of Type 1, or a Type 2 module, with probabilities p_1 and p_2, respectively; where $p_1 + p_2 = 1$. Therefore the rate parameters are given by:

$$p_k f(i,j) \ , \quad k = 1,2$$

where

$$f(i,j) = \lambda \min\{a, M_1-i, M_2-j\}$$

while $g_1(i,j)$ and $g_2(i,j)$ are the same as given by equations (2.17).

Define $\{X_1(t), X_2(t);\ t \geq 0\}$ to be the number of modules of Type 1 and Type 2 down at time t, respectively.

$\{X_1(t), X_2(t),\ t \geq 0\}$ is a bivariate birth-and-death process, operating on the state space $S = \{(0,0),\ldots,(M_1,M_2)\}$. Figure (2.3) shows a graphical representation of the rates of transition between states.

It is clear from the figure (2.3) that state (M_1,M_2) is a transient state. Hence:

$$\lim_{t\to\infty} P_{M_1M_2}(t) = 0$$

It is also clear that once the state vector departs from value (M_1,M_2), the process will never reach (M_1,M_2) again. So unless the process starts in state (M_1,M_2), this state will never be visited. Since the initial condition is assumed to have all modules up, the supervisor can never expect to have all modules of both types down at any point in time. This results from the fact that under this formulation it is assumed that when an aircraft is down awaiting modules, it is waiting for one module only, never both types. This assumption is very optimistic and it doesn't represent a flight-line logistics operation. Therefore, it was concluded that this formulation is inappropriate for the aircraft-squadron logistics problem.

The preceeding two formulations suggest that a simultaneous failure mechanism (i.e., $p_{12} > 0$) is necessary in order to provide operational realism. This is carried out in the next section.

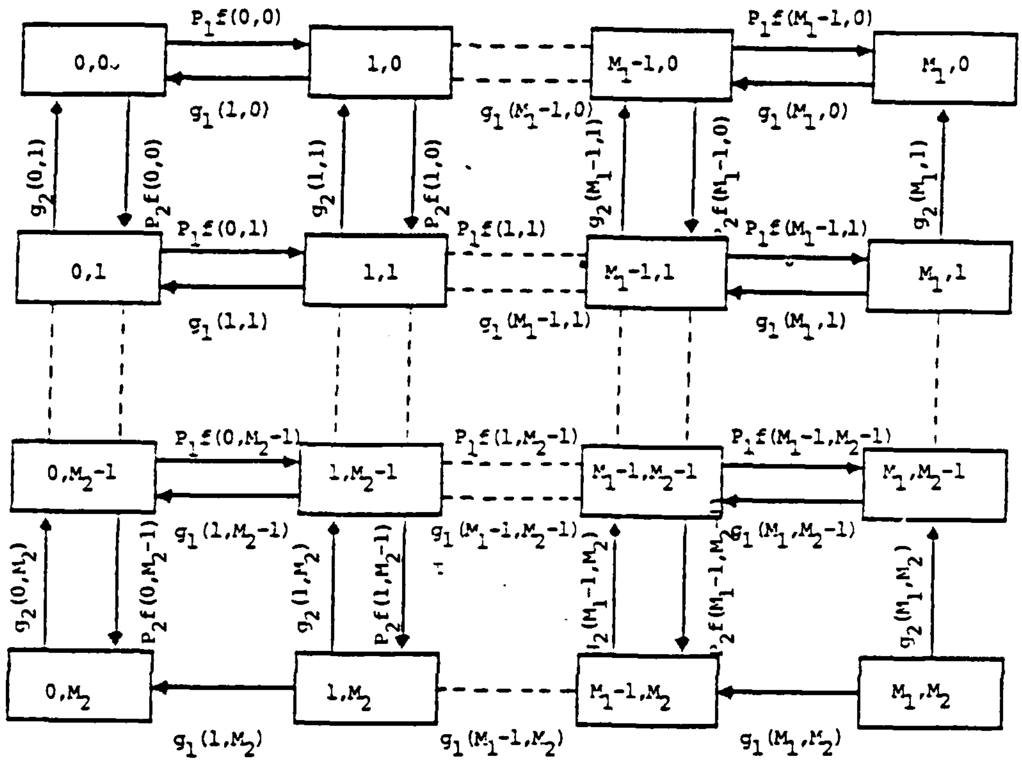

Figure 2.3. Transition Flow for the Two-Modules Logistics Process Under Formulation II

3. Formulation III

Under this formulation, it is assumed that the conditional probability of a simultaneous failure of both Types (p_{12}) is positive. $\{X_1(t), X_2(t),\ t \geq 0\}$ is a bivariate birth-and-death process with parameters given by (2.17).

Define $P_{nm,ij}(t)$ to be the conditional probability of the process being in state (i,j) given that it started in (n,m). Then,

$$P_{nm,ij}(t) = P\{X_1(t) = i, X_2(t) = j \mid X_1(0) = n, X_2(0) = m\}.$$

To simplify the notation we suppress the initial condition and write,

$$P_{ij}(t) = P_{nm,ij}(t) \ .$$

To obtain the forward Chapman-Kolmogorov equations, we write the probability that the process is in state (i,j) at time t+dt as a function of the state probabilities at time t. The probability of being in state (i,j) at time t+dt is expressed by:

$$\begin{aligned} P_{i,j}(t+dt) = \; & (1-(f(i,j) + g_1(i,j) + g_2(i,j)dt)P_{i,j}(t) \\ & + p_1(f(i-1,j)dtP_{i-1,j}(t) + p_{12}f(i-1,j-1)dt \\ & \times P_{i-1,j-1}(t) + p_2f(i,j-1)dtP_{i,j-1}(t) \\ & + g_1(i+1,j)dtP_{i+1,j}(t) + g_2(i,j+1)dtP_{i,j+1}(t) \\ & + o(dt) \end{aligned}$$

Subtracting $P_{i,j}(t)$ from both sides and dividing by dt, and letting dt tend to zero, gives the following:

$$\frac{dP_{0,0}(t)}{dt} = -f(0,0)P_{0,0}(t) + g_1(1,0)P_{1,0}(t) + g_2(0,1)P_{0,1}(t)$$

$$\frac{dP_{i,0}(t)}{dt} = -(f(i,0) + g_1(i,0))P_{i,0}(t) + p_1f(i-1,0)P_{i-1,0}(t)$$
$$+ g_1(i+1,0)P_{i+1,0}(t) + g_2(i,1)P_{i,1}(t)$$
$$1 \leq i \leq M_1-1$$

$$\frac{dP_{0,j}(t)}{dt} = -(f(0,j) + g_2(0,j))P_{0,j}(t) + p_2f(0,j-1)P_{0,j-1}(t)$$
$$+ g_1(1,j)P_{1,j}(t) + g_2(0,j+1)P_{0,j+1}(t)$$
$$1 \leq j \leq M_2-1$$

$$\frac{dP_{i,j}(t)}{dt} = -(f(i,j) + g_1(i,j) + g_2(i,j))P_{i,j}(t)$$
$$+ p_1f(i-1,j)P_{i-1,j}(t) + p_{12}f(i-1,j-1)P_{i-1,j-1}(t)$$
$$+ p_2f(i,j-1)P_{i,j-1}(t) + g_1(i+1,j)P_{i+1,j}(t)$$
$$+ g_2(i,j+1)P_{i,j+1}(t)$$
$$1 \leq i \leq M_1-1, \quad 1 \leq j \leq M_2-1 \qquad (2.21)$$

$$\frac{dP_{i,M_2}(t)}{dt} = -(g_1(i,M_2) + g_2(i,M_2))P_{i,M_2}(t)$$
$$+ p_{12}f(i-1,M_2-1)P_{i-1,M_2-1}(t)$$
$$+ p_2f(i,M_2-1)P_{i,M_2-1}(t)$$
$$+ g_1(i+1,M_2)P_{i+1,M_2}(t) \qquad 1 \leq i \leq M_1-1$$

$$\frac{dP_{M_1,j}(t)}{dt} = -(g_1(M_1,j) + g_2(M_1,j))P_{M_1,j}(t)$$
$$+ p_1 f(M_1-1,j)P_{M_1-1,j}(t)$$
$$+ p_{12} f(M_1-1,j-1)P_{M_1-1,j-1}(t)$$
$$+ g_2(M_1,j+1)P_{M_1,j+1}(t) \qquad 1 \leq j \leq M_2-1$$

$$\frac{dP_{M_1,M_2}(t)}{dt} = -(g_1(M_1,M_2) + g_2(M_1,M_2))P_{M_1,M_2}(t)$$
$$+ p_{12} f(M_1-1,M_2-1)P_{M_1-1,M_2-1}(t)$$

with initial conditions:

$$P_{i,j}(0) = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{otherwise} \end{cases}$$

The bivariate continuous-time-Markov process $\{X_1(t),X_2(t);\ t \geq 0\}$ has already been shown to have a limiting distribution. Define $\pi_{i,j}$ to be the limiting probability of the process being in state (i,j), i.e.,

$$\pi_{i,j} = \lim_{t\to\infty} P_{i,j}(t)$$

Taking the limit, as $t \to \infty$, in the forward Chapman-Kolmogorov equations, gives the following system of equations:

$$-f(0,0)\pi_{00} + g_1(1,0)\pi_{10} + g_2(0,1)\pi_{01} = 0$$

$$-(f(i,0) + g_1(i,0))\pi_{i0} + p_1 f(i-1,0)\pi_{i-1,0}$$
$$+ g_1(i+1,0)\pi_{i+1,0} + g_2(i,1)\pi_{i1}) = 0$$

$$-(f(0,j) + g_2(0,j))\pi_{0j} + p_2 f(0,j-1)\pi_{0,j-1}$$
$$+ g_1(1,j)\pi_{1j} + g_2(0,j+1)\pi_{0,j+1} = 0$$

$$-(f(i,j) + g_1(i,j) + g_2(i,j))\pi_{ij} + p_1 f(i-1,j)\pi_{i-1,j}$$
$$+ p_{12} f(i-1,j-1)\pi_{i-1,j-1} + p_2 f(i,j-1)\pi_{i,j-1}$$
$$+ g_1(i+1,j)\pi_{i+1,j} + g_2(i,j+1)\pi_{i,j+1} = 0 \qquad (2.22)$$

$$-(g_1(i,M_2) + g_2(i,M_2))\pi_{i,M_2} + p_{12} f(i-1,M_2-1)\pi_{i-1,M_2-1}$$
$$+ p_2 f(i,M_2-1)\pi_{i,M_2-1} + g_1(i+1,M_2)\pi_{i+1,M_2} = 0$$

$$-(g_1(M_1,j) + g_2(M_1,j)\pi_{M_1,j} + p_1 f(M_1-1,j)\pi_{M_1-1,j}$$
$$+ p_{12} f(M_1-1,j-1)\pi_{M_1-1,j-1} + g_2(M_1,j+1)\pi_{M_1,j+1} = 0$$

$$-(g_1(M_1+M_2) + g_2(M_1,M_2))\pi_{M_1,M_2} + p_{12} f(M_1-1,M_2-1)\pi_{M_1-1,M_2-1} = 0$$

$$\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} \pi_{ij} = 1$$

Therefore, it is required to solve $(M_1+1)(M_2+1)$ linear systems of equations to compute π_{ij} $\forall$ $(i,j) \in S$. The size of the

system of equations tends to be large as M_1 and M_2 increase. For example, if there are 50 modules of both types, the system has 2,601 equations. Thus, for a large maintenance system, the above approach would be infeasible to follow. Therefore, another approach is desirable. The Matrix Geometric Method of M. Neuts (1981) is the alternative pursued here.

H. MATRIX GEOMETRIC APPROACH

Following the approaches in Neuts (1981), and Gani and Purdue (1984), we now formulate the continuous time Markov process model for the two-module logistics problem. This formulation facilitates computations for modules of realistic number.

Consider $\{X_1(t), X_2(t);\ t \geq 0\}$ as a Markov process on the state space $\{S_{M_1}, S_{M_1-1}, \ldots, S_1, S_0\}$ defined below.

$$S_{M_1} = \{(M_1,0),(M_1,1),\ldots,(M_1,M_2)\}$$

$$S_{M_1-1} = \{(M_1-1,0),(M_1-1,1),\ldots,(M_1-1,M_2)\}$$

$$\vdots$$

$$S_1 = \{(1,0),(1,1),\ldots,(1,M_2)\}$$

$$S_0 = \{(0,0),(0,1),\ldots,(0,M_2)\}$$

Let

$$\vec{P}_{M_1}(t) = \{P_{M_1,0}(t), P_{M_1,1}(t), \ldots, P_{M_1,M_2}(t)\}$$

$$\vec{P}_{M_1-1}(t) = \{P_{M_1-1,0}(t), P_{M_1-1,1}(t), \ldots, P_{M_1-1,M_2}(t)\}$$

$$\vdots$$

$$\vec{P}_1(t) = \{P_{1,0}(t), P_{1,1}(t), \ldots, P_{1,M_2}(t)\}$$

$$\vec{P}_0(t) = \{P_{0,0}(t), P_{0,1}(t), \ldots, P_{0,M_2}(t)\}$$

Define the following matrices:

$$F^{(n)} = \begin{bmatrix} f(n,0) & & & & \\ & f(n,1) & & & \\ & & \ddots & & \\ & & & f(n,M_2-1) & \\ & & & & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

$$G_1^{(n)} = \begin{bmatrix} g_1(n,0) & & & \\ & g_1(n,1) & & \\ & & \ddots & \\ & & & g_1(n,M_2) \end{bmatrix}$$

$$G_2^{(n)} = \begin{bmatrix} 0 & & & \\ & g_2(n,1) & & \\ & & \ddots & \\ & & & g_2(n,M_2) \end{bmatrix}$$

Let Q be the infinitesimal generator for the Markov process $\{X_1(t), X_2(t);\ t \geq 0\}$. Then Q will be a block tri-diagonal matrix as follows:

$$Q = \begin{bmatrix} B_0 & C_0 & & & & \\ D_1 & B_1 & C_1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & 0 & & D_{M_1-1} & B_{M_1-1} & C_{M_1-1} \\ & & & & D_{M_1} & B_{M_1} \end{bmatrix}$$

where B_n, C_n, and D_n are square matrices of order M_2+1. The diagonal elements of the diagonal matrices B_n are strictly negative and are determined so that the row sums of Q are zero. The process can be fully defined by the infinitesimal generator and the initial condition. Therefore, we need to determine all of the elements of B_n, C_n and D_n.

The forward Chapman-Kolmogorov equations are given by:

$$\frac{d}{dt}\vec{P}(t) = \vec{P}(t)Q$$

i.e.,

$$\frac{d}{dt}[\vec{P}_0(t)\ \vec{P}_1(t)\ \dots\ \vec{P}_{M_1}(t)] = [\vec{P}_0(t)\ \vec{P}_1(t)\ \dots\ \vec{P}_{M_1}(t)]$$

$$\times \begin{bmatrix} B_0 & C_0 & & & \\ D_1 & B_1 & C_1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & & & \\ & D_{M_1-1} & B_{M_1-1} & C_{M_1-1} & \\ & & D_{M_1} & B_{M_1} & \end{bmatrix}$$

Hence, the forward Chapman-Kolmogorov equations become:

$$\frac{d}{dt}\vec{P}_0(t) = \vec{P}_0(t)B_0 + \vec{P}_1(t)D_1$$

$$\frac{d}{dt}\vec{P}_n(t) = \vec{P}_{n-1}(t)C_{n-1} + \vec{P}_n(t)B_n + \vec{P}_{n+1}(t)D_{n+1} \qquad (2.23)$$

$$1 \leq n \leq M_1-1$$

$$\frac{d}{dt}\vec{P}_{M_1}(t) = \vec{P}_{M_1-1}(t)C_{M_1-1} + \vec{P}_{M_1}(t)B_{M_1}$$

From equations (2.21), we can obtain the equations for $\frac{d}{dt}\vec{P}_n(t)$, $(n = 0,\ldots,M_1)$, and expressing these in matrix form gives:

$$\frac{d}{dt}\vec{P}_{M_1}(t) = [P_{M_1,0}(t) \;\ldots\; P_{M_1,M_2}(t)]$$

$$\times \begin{bmatrix} -g_1(M_1,0) & & & & 0 \\ g_2(M_1,1) & -(g_1(M_1,1)+g_2(M_1,1)) & & & \\ & \ddots & \ddots & & \\ 0 & & & & \\ & & & g_2(M_1,M_2) & -(g_1(M_1,M_2)+g_2(M_1,M_2)) \end{bmatrix}$$

$$+ [P_{M_1-1,0}(t) \;\ldots\; P_{M_1-1,M_2}(t)]$$

$$\times \begin{bmatrix} p_1 f(M_1-1,0) & p_{12} f(M_1-1,0) & & & \\ & p_1 f(M_1-1,1) & p_{12} f(M_1-1,1) & & \\ & & \ddots & \ddots & \\ & & & p_1 f(M_1-1,M_2-1) & p_{12} f(M_1-1,M_2-1) \\ & & & & 0 \end{bmatrix}$$

$$
\frac{d}{dt}\vec{P}_n(t) = [P_{n,0}(t) \;\cdots\; P_{n,M_2}(t)]
$$

$$
\times \begin{bmatrix}
-(f(n,0)+g_1(n,0)) & p_2 f(n,0) & & & \\
g_2(n,1) & -(f(n,1)+g_1(n,1)+g_2(n,1)) & p_2 f(n,1) & & \\
& \ddots & \ddots & \ddots & \\
& & g_2(n,M_2-1) & -(f(n,M_2-1)+g_1(n,M_2-1)+g_2(n,M_2-1)) & p_2 f(n,M_2-1) \\
& & & g_2(n,M_2) & -(g_1(n,M_2)+g_2(n,M_2))
\end{bmatrix}
$$

$$
+ [P_{n+1,0}(t) \;\cdots\; P_{n+1,M_2}(t)] \begin{bmatrix}
g_1(n,0) & & & 0 \\
& g_1(n,1) & & \\
& & \ddots & \\
0 & & & g_1(n,M_2)
\end{bmatrix}
$$

$$
+ [P_{n-1,0}(t) \;\cdots\; P_{n-1,M_2}(t)]
$$

$$
\times \begin{bmatrix}
p_1 f(n-1,0) & p_{12} f(n-1,0) & & & \\
& p_1 f(n-1,1) & p_{12} f(n-1,1) & & \\
& & \ddots & \ddots & \\
& & & p_1 f(n-1,M_2-1) & p_{12} f(n-1,M_2-1) \\
& & & & 0
\end{bmatrix}
$$

for $n = 1,\ldots,M_1-1$

$$\frac{d}{dt}\vec{P}_0(t) = [P_{0,0}(t) \ \ldots \ P_{0,M_2}(t)]$$

$$\times \begin{bmatrix} -f(0,0) & p_2 f(0,0) & & & \\ g_2(0,1) & -(f(0,1)+g_2(0,1)) & p_2 f(0,1) & & \\ & \ddots & \ddots & \ddots & \\ & & g_2(0,M_2-1) & -(f(0,M_2-1)+g_2(0,M_2-1)) & p_2 f(0,M_2-1) \\ & & & g_2(0,M_2) & -g_2(0,M_2) \end{bmatrix}$$

$$+ [P_{1,0}(t) \ \ldots \ P_{1,M_2}(t)] \begin{bmatrix} g_1(1,0) & & & \\ & g_1(1,1) & & \\ & & \ddots & \\ & & & g_1(1,M_2) \end{bmatrix}$$

Therefore we get the following set of equations:

$$\frac{d}{dt}\vec{P}_0(t) = \vec{P}_0(t)[-(F^{(0)}+G_2^{(0)})+G_2^{(0)}A^T+p_2F^{(0)}A] + \vec{P}_1(t)G_1^{(1)}$$

$$\begin{aligned} \frac{d}{dt}\vec{P}_n(t) = {} & \vec{P}_{n-1}(t)[p_1F^{(n-1)}+p_{12}F^{(n-1)}A] \\ & + \vec{P}_n(t)[-(F^{(n)}+G_1^{(n)}+G_2^{(n)})+G_2^{(n)}A^T+p_2F^{(n)}A] \\ & + \vec{P}_{n+1}(t)G_1^{(n+1)} \end{aligned} \tag{2.24}$$

$$\begin{aligned} \frac{d}{dt}\vec{P}_{M_1}(t) = {} & \vec{P}_{M_1-1}(t)[p_1F^{(M_1-1)}+p_{12}F^{(M_1-1)}A] \\ & + \vec{P}_{M_1}(t)[-(G_1^{(M_1)}+G_2^{(M_2)})+G_2^{(M_1)}A^T] \end{aligned}$$

with initial condition:

$$\vec{P}_0(0) = [1 \quad 0 \quad \ldots \quad 0] \,, \qquad \vec{P}_n(0) = 0 \qquad 1 \le n \le M_1$$

Thus, by comparing the system of equations (2.24) to (2.23), we have the following result.

Proposition (2.1)

The Bivariate-Birth-and-Death process $\{X_1(t), X_2(t);\ t \ge 0\}$ for the Two-Module-Logistics model with simultaneous failure has infinitesimal generator

$$Q = \begin{bmatrix} B_0 & C_0 & & & \\ D_1 & B_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & D_{M_1-1} & B_{M_1-1} & C_{M_1-1} \\ & & & D_{M_1} & B_{M_1} \end{bmatrix}$$

where:

$$B_0 = [-(F^{(0)}+G_2^{(0)}) + G_2^{(0)}A^T + p_2F^{(0)}A]$$

$$B_n = [-(F^{(n)}+G_1^{(n)}+G_2^{(n)}) + G_2^{(n)}A^T + p_2F^{(n)}A] \qquad 1 \le n \le M_1-1$$

$$B_{M_1} = [-(G_1^{(M_1)}+G_2^{(M_1)}) + G_2^{(M_1)}A^T]$$

$$C_n = [p_1F^{(n)} + p_{12}F^{(n)}A] \qquad 0 \le n \le M_1-1$$

$$D_n = G_1^{(n)} \qquad 1 \le n \le M_1$$

Therefore, by knowing $F^{(n)}$, $G_1^{(n)}$ and $G_2^{(n)}$ $(n = 0,1,\ldots,M_1)$, we can use the above proposition to construct the infinitesimal generator, and hence the Chapman-Kolmogorov equations. Now, we are ready to find the limiting distribution for the process by adapting the argument from Gaver, Jacobs and Latouche (1984).

1. The Limiting Distribution

The Markov-process $\{X_1(t), X_2(t);\ t \ge 0\}$ has been shown to have a limiting distribution. Let

$$\vec{\pi}_n = \lim_{t\to\infty} \vec{P}_n(t)$$

as $t \to \infty$, we get:

$$\lim_{t\to\infty} \frac{d}{dt} \vec{P}_n(t) = \vec{0}$$

Therefore, from (2.23), it follows that:

$$\vec{\pi}_0 B_0 + \vec{\pi}_1 D_1 = \vec{0}$$

$$\vec{\pi}_{n-1} C_{n-1} + \vec{\pi}_n B_n + \vec{\pi}_{n+1} D_{n+1} = \vec{0} \qquad 1 \le n \le M_1 - 1 \tag{2.25}$$

$$\vec{\pi}_{M_1-1} C_{M_1-1} + \vec{\pi}_{M_1} B_{M_1} = \vec{0}$$

From the third line in (2.20), we get:

$$\vec{\pi}_{M_1} B_{M_1} = -\vec{\pi}_{M_1-1} C_{M_1-1}$$

Thus:

$$\vec{\pi}_{M_1} = -\vec{\pi}_{M_1-1} C_{M_1-1} (B_{M_1})^{-1} = \vec{\pi}_{M_1-1} C_{M_1-1} (-H_{M_1}^{-1}) \qquad (2.26)$$

From the second line in (2.25), when $n = M_1-1$, we get:

$$\vec{\pi}_{M_1-2} C_{M_1-2} + \vec{\pi}_{M_1-1} B_{M_1-1} + \vec{\pi}_{M_1} D_{M_1} = \vec{0}$$

Substituting (2.26), we obtain:

$$\vec{\pi}_{M_1-2} C_{M_1-2} + \vec{\pi}_{M_1-1} B_{M_1-1} + \vec{\pi}_{M_1-1} C_{M_1-1} (-H_{M_1}^{-1}) D_{M_1} = \vec{0}$$

Therefore:

$$\vec{\pi}_{M_1-1} [B_{M_1-1} + C_{M_1-1} (-H_{M_1}^{-1}) D_{M_1}] = -\vec{\pi}_{M_1-2} C_{M_1-2}$$

$$\vec{\pi}_{M_1-1} = \vec{\pi}_{M_1-2} C_{M_1-2} (-H_{M_1-1}^{-1})$$

where:

$$H_{M_1-1} = B_{M_1-1} + C_{M_1-1} (-H_{M_1}^{-1}) D_{M_1}$$

The claim is that:

$$\vec{\pi}_n = \vec{\pi}_{n-1} C_{n-1} (-H_n^{-1})$$

where:

$$H_n = B_n + C_n (-H_{n+1}^{-1}) D_{n+1}$$

$$H_{M_1} = B_{M_1}$$

We will prove this by mathematical induction. We know from equation (2.26) that the statement is true for $n = M_1$. Suppose it is true for $n = i$. We need to show that the statement is true for $n = i-1$. From the second line of (2.20), we get:

$$\vec{\pi}_{i-2} C_{i-2} + \vec{\pi}_{i-1} B_{i-1} + \vec{\pi}_i D_i = \vec{0}$$

Therefore:

$$\vec{\pi}_{i-2} C_{i-2} + \vec{\pi}_{i-1} B_{i-1} + \vec{\pi}_{i-1} C_{i-1} (-H_i^{-1}) D_i = \vec{0}$$

$$\vec{\pi}_{i-1} [B_{i-1} + C_{i-1} (-H_i^{-1}) D_i] = -\vec{\pi}_{i-2} C_{i-2}$$

$$\vec{\pi}_{i-1} = -\vec{\pi}_{i-2} C_{i-2} [-H_{i-1}^{-1}]$$

which proves the statement is true for $n = i-1$.

In the case of n = 0, we have:

$$\vec{\pi}_0 B_0 + \vec{\pi}_1 D_1 = \vec{0}$$

$$\vec{\pi}_0 B_0 + \vec{\pi}_0 C_0 [-H_1^{-1}] D_1 = \vec{0}$$

Therefore:

$$\vec{\pi}_0 H_0 = \vec{0}$$

Thus, we have proved the proposition.

Proposition (2.2)

The limiting distribution for the bivariate-Birth-and-Death process, for the Two Module Logistics model with simultaneous failure is given by the following recursive equations:

$$\vec{\pi}_0 H_0 = \vec{0}$$

$$\vec{\pi}_n = \vec{\pi}_{n-1} C_{n-1} (-H_n^{-1}) \qquad 1 \le n \le M_1$$

$$\sum_{n=0}^{M_1} \vec{\pi}_n e = 1$$

where:

$$H_{M_1} = B_{M_1}$$

$$H_n = B_n + C_n (-H_{n+1}^{-1}) D_{n+1} \qquad 0 \le n \le M_1 - 1$$

$$e^T = [1 \;\; 1 \;\; \ldots \;\; 1]$$

and B_n, C_n, and D_n are as given by Proposition (2.1).

2. Algorithm for Computing the Limiting Distribution

The following algorithm was used to solve the recursive equations given by Proposition (2.2). The algorithm is based on the one given in Gaver, Jacobs and Latouche (1984).

1. Starting with $H_{M_1} = B_{M_1}$, compute recursively the matrices H_n, for $0 \leq n \leq M_1-1$.
2. Solve the system $\vec{\pi}_0 H_0 = 0$, $\vec{\pi}_0 e = 1$.
3. Compute recursively the vectors $\vec{\pi}_n$, $n = 1,2,\ldots,M_1$; and renormalize the vectors $\vec{\pi}$'s so obtained at each stage (i.e., renormalize each time a new subvector is determined).

3. The Laplace Transforms

Following the argument in Gani and Purdue (1984) we can obtain an expression for $\hat{\vec{P}}_n(s)$, the Laplace transform of $\vec{P}_n(t)$.

$$\hat{\vec{P}}_n(s) = \int_0^\infty e^{-st} \vec{P}_n(t)\,dt$$

Multiplying both sides of (2.23) by e^{-st} and integrating over all values of t, we get the following:

$$s\vec{\hat{P}}_0(s) - e_0 = \vec{\hat{P}}_0(s)B_0 + \vec{\hat{P}}_1(s)D_1$$

$$s\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{n-1}(s)C_{n-1} + \vec{\hat{P}}_n(s)B_n + \vec{\hat{P}}_{n+1}(s)D_{n+1} \qquad (2.27)$$

$$s\vec{\hat{P}}_{M_1}(s) = \vec{\hat{P}}_{M_1-1}(s)C_{M_1-1} + \vec{\hat{P}}_{M_1}(s)B_{M_1}$$

where e_0 represents the initial condition

$$\vec{\hat{P}}_0(0) = (1 \quad 0 \quad \ldots \quad 0) = e_0$$

Thus, we obtain:

$$\vec{\hat{P}}_{M_1}(s)[sI-B_{M_1}] = \vec{\hat{P}}_{M_1-1}(s)C_{M_1-1}$$

$$\vec{\hat{P}}_{M_1}(s) = \vec{\hat{P}}_{M_1-1}(s)C_{M_1-1}[-E_{M_1}^{-1}] \qquad (2.28)$$

where:

$$E_{M_1} = B_{M_1} - sI$$

We claim that

$$\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{n-1}(s)C_{n-1}[-E_n^{-1}]$$

where:

$$E_n = B_n + C_n[-E_{n+1}^{-1}]D_{n+1} - sI \qquad 0 \le n \le M_1-1$$

$$E_{M_1} = B_{M_1} - sI$$

We have shown (equation 2.28) that the statement is true for $n = M_1$. We will again prove the claim by induction. Suppose that the statement is true for $n = i$. We will show that it must also be true for $n = i-1$.

For $n = i-1$, we get, from equation (2.22) that

$$s\hat{\vec{P}}_{i-1}(s) = \hat{\vec{P}}_{i-2}(s)C_{i-2} + \hat{\vec{P}}_{i-1}(s)B_{i-1} + \hat{\vec{P}}_i(s)D_i$$

But from the induction hypothesis,

$$\hat{\vec{P}}_i(s) = \hat{\vec{P}}_{i-1}(s)C_{i-1}[-E_i^{-1}]$$

Therefore:

$$s\hat{\vec{P}}_{i-1}(s) = \hat{\vec{P}}_{i-2}(s)C_{i-2} + \hat{\vec{P}}_{i-1}(s)B_{i-1} + \hat{\vec{P}}_{i-1}(s)C_{i-1}[-E_i^{-1}]D_i$$

$$\hat{\vec{P}}_{i-1}(s)[-(B_{i-1}+C_{i-1}[-E_i^{-1}]D_i-sI)] = \hat{\vec{P}}_{i-2}(s)C_{i-2}$$

Hence:

$$\hat{\vec{P}}_{i-1}(s) = \hat{\vec{P}}_{i-2}(s)C_{i-2}[-E_{i-1}^{-1}]$$

In the case n = 0, we obtain from the induction hypothesis,

$$\hat{\vec{P}}_1(s) = \hat{\vec{P}}_0(s)C_0[-E_1^{-1}]$$

Substituting into the first line of (2.27), we get:

$$s\hat{\vec{P}}_0(s) - e_0 = \hat{\vec{P}}_0(s)B_0 + \hat{\vec{P}}_0(s)C_0[-E_1^{-1}]D_1$$

$$\hat{\vec{P}}_0(s)[-(B_0+C_0[-E_1^{-1}]D_1-sI] = e_0$$

$$\hat{\vec{P}}_0(s)[-E_0] = e_0$$

We summarize these results as Proposition 2.3.

<u>Proposition (2.3)</u>

The vectors $\hat{\vec{P}}_n(s)$, $n = 0,1,\ldots,M_1$, are determined by the equations:

$$\hat{\vec{P}}_0(s)[-E_0] = e_0$$

$$\hat{\vec{P}}_n(s) = \hat{\vec{P}}_{n-1}(s)C_{n-1}[-E_n^{-1}]$$

where:

$$E_{M_1} = B_{M_1}$$

$$E_n = B_n + C_n[-E_{n+1}^{-1}]D_{n+1} - sI$$

$$e_0 = (1 \quad 0 \quad \ldots \quad 0)$$

and B_n, C_n, and D_n are as given by Proposition (2.1).

Proposition (2.3) can also be used to find the limiting distribution as follows:

$$\vec{\pi}_n = \lim_{t \to \infty} \vec{P}_n(t) = \lim_{s \to 0} s\hat{\vec{P}}_n(s)$$

Now, multiplying the equations for $\hat{\vec{P}}_n(s)$ given by Proposition (2.3) by s and letting $s \to 0$, we obtain:

$$\lim_{s \to 0} s\hat{\vec{P}}_0(s)[-E_0] = \lim_{s \to 0} se_0 \ ,$$

and

$$\lim_{s \to 0} s\hat{\vec{P}}_0(s) = \lim_{s \to 0} s\hat{\vec{P}}_{n-1}(s)C_{n-1}[-E_n^{-1}]$$

But:

$$\lim_{s \to 0} [E_n] = H_n \ .$$

Therefore we get:

$$\vec{\pi}_0[-H_0] = \vec{0}$$

$$\vec{\pi}_n = \vec{\pi}_{n-1}C_{n-1}[-H_n^{-1}] \ .$$

Multiplying the first line in the above equations by -1, and adding the normalizing equation, we obtain the equations for the vectors $\vec{\pi}_n$, $n = 0,1,\ldots,M_1$, given by Proposition (2.2).

Proposition (2.3) can only be applied if the process started with probability 1 in state (0,0). However, the analyst may need to calculate the Laplace transforms for the process with different initial conditions. In particular, suppose one has positive probabilities $P\{X_1(0) = i, X_2(0) = j\}$ for all states.

We need to generalize the above results to allow for other starting conditions, in particular to allow consideration of the partial surprise scenario.

Given the process started in state (i,j),

$$\vec{P}_i(0) = e_j \qquad i = 0,1,\ldots,M_2$$

and

$$\vec{P}_k(0) = \vec{0} \qquad \text{for all other } k,$$

where:

$$e_j = (0 \;\; 0 \;\; \ldots 0 \, 1 \, 0 \ldots \;\; 0) \qquad j = 0,1,\ldots,M_2 \;.$$

The i^{th} element of e_j is equal to 1 whenever the process starts in state (i,j).

For the vectors from $n = M_1$ down to $n = i+1$ of $\vec{\hat{P}}_n(s)$, we can use the results from Proposition (2.3).

$$\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{n-1}(s)C_{n-1}[-E_n^{-1}] \quad \text{for } n = i+1,\ldots,M_2$$

Now, applying the argument in Gaver, Jacobs, and Latouche (1984) to equations (2.23), we obtain:

$$s\vec{\hat{P}}_i(s) - e_j = \vec{\hat{P}}_{i-1}(s)C_{i-1} + \vec{\hat{P}}_i(s)B_i + \vec{\hat{P}}_{i+1}(s)D_{i+1}$$

Therefore:

$$\vec{\hat{P}}_i(s) = \vec{\hat{P}}_{i-1}(s)C_{i-1}[-E_i^{-1}] + e_j[-E_i^{-1}] \tag{2.29}$$

Mathematical induction can be used to show:

$$\vec{\hat{P}}_{i-k}(s) = \vec{\hat{P}}_{i-k-1}(s)C_{i-k-1}[-E_{i-k}^{-1}]$$
$$+ e_j[-E_i^{-1}] \prod_{\ell=1}^{k} D_{i-\ell+1}[-E_{i-\ell}^{-1}]$$
$$\text{for } 0 \le k \le i$$

To show that the statement is true for $k = 1$, note that from equations (2.23), we obtain:

$$s\vec{\hat{P}}_{i-1}(s) = \vec{\hat{P}}_{i-2}(s)C_{i-2} + \vec{\hat{P}}_{i-1}(s)B_{i-1} + \vec{\hat{P}}_i(s)D_i$$

From (2.29) we obtain:

$$s\hat{\vec{P}}_{i-1}(s) = \hat{\vec{P}}_{i-2}(s)C_{i-2} + \hat{\vec{P}}_{i-1}(s)B_{i-1} + \hat{\vec{P}}_{i-1}(s)C_{i-1}[-E_i^{-1}]D_i$$
$$+ e_j[-E_i^{-1}]D_i$$

$$\hat{\vec{P}}_{i-1}(s) = \hat{\vec{P}}_{i-2}(s)C_{i-2}[-E_{i-1}^{-1}] + e_j[-E_i^{-1}]D_i[-E_{i-1}^{-1}] \ .$$

Therefore the statement is true for k = 1. Suppose it is true for $k = m$, $0 < m < i$. We need to show that it is true for $k = m+1$. From the induction hypothesis we have,

$$\hat{\vec{P}}_{i-m}(s) = \hat{\vec{P}}_{i-m-1}(s)C_{i-m-1}[-E_{i-m}^{-1}] + e_j[-E_i^{-1}]\prod_{\ell=1}^{m} D_{i-\ell+1}[-E_{i-\ell}^{-1}]$$

For $k = m+1$, using equation (2.18),

$$s\hat{\vec{P}}_{i-m-1}(s) = \hat{\vec{P}}_{i-m-2}(s)C_{i-m-2} + \hat{\vec{P}}_{i-m-1}(s)B_{i-m-1} + \hat{\vec{P}}_{i-m}(s)D_{i-m}$$
$$= \hat{\vec{P}}_{i-m-2}(s)C_{i-m-2} + \hat{\vec{P}}_{i-m-1}(s)B_{i-m-1}$$
$$+ \hat{\vec{P}}_{i-m-1}(s)e_{i-m-1}[-E_{i-m}^{-1}]D_{i-m}$$
$$+ e_j[-E_i^{-1}] \prod_{\ell=1}^{m} D_{i-\ell+1}[-E_{i-\ell}^{-1}]D_{i-m}$$

Therefore:

$$\vec{P}_{i-m-1}(s) = \vec{\hat{P}}_{i-m-2}(s)C_{i-m-2}[-E^{-1}_{i-m-1}]$$

$$+ e_j[-E_i^{-1}] \prod_{\ell=1}^{m+1} D_{i-\ell+1}[E^{-1}_{i-\ell}]$$

We have proved the following proposition.

<u>Proposition (2.4)</u>

The vectors $\vec{\hat{P}}_n(s)$, $n = 0,1,\ldots,M_1$, given that the process started in state (i,j), are given by:

$$\vec{\hat{P}}_0(s) = e_j[-E_i^{-1}] \prod_{\ell=1}^{i} D_{i-\ell+1}[-E^{-1}_{i-\ell}]$$

$$\vec{\hat{P}}_{i-k}(s) = \vec{\hat{P}}_{i-k-1}(s)C_{i-k-1}[-E^{-1}_{i-k}]$$

$$+ e_j[-E_i^{-1}] \prod_{\ell=1}^{k} D_{i-\ell+1}[-E^{-1}_{i-\ell}]$$

$$0 \le k \le i-1; \quad i \le n \le M_1$$

$$\vec{P}_n(s) = \vec{\hat{P}}_{n-1}(s)C_{n-1}[-E_n^{-1}]$$

The readiness distribution can now be computed.

a. Total Surprise.

Under the Two-Module Logistics formulation, with a total surprise scenario, a readiness distribution model can be derived as follows:

$$P\{D(0) = k\} = \begin{cases} \sum_{i=0}^{M_1-k} \pi_{i,M_2-k} + \sum_{j=0}^{M_2-k-1} \pi_{M_1-k,j} & 0 \leq k \leq a-1 \\ \sum_{i=0}^{M_1-a} \sum_{j=0}^{M_2-a} \pi_{i,j} & k = a \\ 0 & \text{otherwise} \end{cases} \quad (2.30)$$

This assumes that attacks occur infrequently, so a steady-state distribution prevails, and that there is essentially no warning.

b. Partial Surprise

As in the Single-Module model formulation, suppose that the intelligence data suggests that the enemy will strike after W units of time (e.g., days). Let W be a random variable exponentially distributed with mean η^{-1}. Assume that during the time W the failure rate decreases from λ to λ' where $\lambda > \lambda'$, and repair rates increase from μ_1 and μ_2 to μ_1' and μ_2', respectively, where $\mu_1 < \mu_1'$ and $\mu_2 < \mu_2'$.

Let $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$ be the number of modules of Type 1 and Type 2 down at time t. The process $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$ is a bivariate-continuous-time-Markov process with rate parameters:

$$\tilde{f}(i,j) = \lambda' \min\{a, M_1-i, M_2-j\}$$

$$\tilde{g}_1(i,j) = \begin{cases} \mu_1' i & i = 1,2,\ldots,R_1-1 \\ \mu_1' R_1 & i = R_1,\ldots,M_1 \end{cases} \tag{2.31}$$

$$\tilde{g}_2(i,j) = \begin{cases} \mu_2' j & j = 1,2,\ldots,R_2-1 \\ \mu_2' R_2 & j = R_2,\ldots,M_2 \end{cases}$$

The initial distribution for $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$ is the limiting distribution for $\{X_1(t), X_2(t);\ t \geq 0\}$ with parameters λ, μ_1, and μ_2, given by Proposition (2.2):

$$P\{\tilde{X}_1(0) = i, \tilde{X}_2(0) = j\} = \pi_{i,j} \quad (i,j) \in S \tag{2.32}$$

Therefore, $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$ is a bivariate birth-and-death process, with initial conditions $\pi_{i,j}$ for $(i,j) \in S$. Let

$$\tilde{P}_{nm,ij}(t) = P\{\tilde{X}_1(t) = i, \tilde{X}_2(t) = j \mid \tilde{X}_1(0) = n, \tilde{X}_2(0) = m\}$$

and $\hat{\tilde{P}}_{nm,ij}(s)$ denote the Laplace transform for $\tilde{P}_{nm,ij}(t)$. Then,

$$\hat{\tilde{P}}_{nm,ij}(s) = \int_0^\infty e^{-st}\, \tilde{P}_{nm,ij}(t)\,dt$$

To find the readiness probability distribution at the time of attack, it is required to find the number of modules of both types down at the time of attack.

Let $\tilde{P}_{nm,ij}(W)$ be the conditional probability of the process being in state (i,j) at W (i.e., when the attack occurs) given the process was in state (n,m) initially, and that W is exponentially distributed with mean η^{-1}. Therefore,

$$\tilde{P}_{nm,ij}(W) = P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j \mid \tilde{X}_1(0) = n, \tilde{X}_2(0) = m, W\}$$

Now, note that

$$E[\tilde{P}_{nm,ij}(W)] = \int_0^\infty \tilde{P}_{nm,ij}(t)\, \eta e^{-\eta t} dt$$

It follows that:

$$E[\tilde{P}_{nm,ij}(W)] = \eta \hat{\tilde{P}}_{nm,ij}(\eta)$$

So, as in the case of the single-module model, the required readiness is given <u>in terms of</u> the Laplace transforms.

In much of what follows, for simplicity of notation, we will suppress the initial condition and write,

$$\tilde{P}_{ij}(t) = \tilde{P}_{nm,ij}(t) ,$$

$$\hat{\tilde{P}}_{ij}(s) = \hat{\tilde{P}}_{nm,ij}(s) ,$$

and

$$\tilde{P}_{ij}(W) = \tilde{P}_{nm,ij}(W) .$$

Let

$$\hat{\vec{P}}_{\sim n}(s) = [\hat{\tilde{P}}_{n0}(s), \hat{\tilde{P}}_{n1}(s), \ldots, \hat{\tilde{P}}_{nM_2}(s)] .$$

Using Proposition (2.4), we find that the vectors for the Laplace transforms for the new, restricted, bivariate birth-and-death process, $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$, $\hat{\vec{P}}_{\sim n}(s)$, $n = 0,1, \ldots, M_1$, can be determined, where:

$$F^{(n)} = \begin{bmatrix} \tilde{f}(n,0) & & & & \\ & \tilde{f}(n,1) & & & \\ & & \ddots & & \\ & & & \tilde{f}(n,M_2-1) & \\ & & & & 0 \end{bmatrix}$$

$$G_1^{(n)} = \begin{bmatrix} \tilde{g}_1(n,0) & & & \\ & \tilde{g}_1(n,1) & & \\ & & \ddots & \\ & & & \tilde{g}_1(n,M_2) \end{bmatrix}$$

$$G_2^{(n)} = \begin{bmatrix} 0 & & & \\ & \tilde{g}_2(n,1) & & \\ & & \ddots & \\ & & & \tilde{g}_2(n,M_2) \end{bmatrix}$$

The readiness probability distribution can be calculated by solving for $P\{\tilde{X}(W) = i, \tilde{X}(W) = j | W\}$.

From the law of total probability

$$P\{\tilde{X}(W) = i, \tilde{X}(W) = j | W\} = \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j, \tilde{X}_1(0) = n, \tilde{X}_2(0) = m | W\}$$

$$= \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j | \tilde{X}_1(0) = n, \tilde{X}_2(0) = m, W\} \times P\{\tilde{X}_1(0) = n, \tilde{X}_2(0) = m\}$$

Therefore:

$$P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j | W\} = \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j | \tilde{X}_1(0) = n, \tilde{X}_2(0) = m, W\} \pi_{n,m} \tag{2.33}$$

Since $\pi_{n,m}$ is non-random, taking the expectations of both sides of (2.33) with respect to W, gives the following:

$$E[P\{\tilde{X}_1(W) = i, \tilde{X}_2(W) = j \mid W\}] = \eta \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} \pi_{n,m} \hat{\tilde{P}}_{nm,ij}(\eta) \qquad (2.35)$$

Hence, the readiness probability distribution is then given by:

$$P\{D(0) = k\} = \begin{cases} \sum_{i=0}^{M_1-k} E[\tilde{P}_{i,M_2-k}(W)] + \sum_{j=0}^{M_2-k-1} E[\tilde{P}_{M_1-k,j}(W)] & 0 \le k \le a-1 \\ \sum_{i=0}^{M_1-a} \sum_{j=0}^{M_2-a} E[\tilde{P}_{ij}(W)] & k=a \\ 0 & \text{otherwise} \end{cases} \qquad (2.36)$$

In terms of the Laplace transforms the above becomes:

$$P\{D(0) = k\} = \begin{cases} \eta \sum_{i=0}^{M_1-k} \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} \pi_{n,m} \hat{\tilde{P}}_{n,m;i,M_2-k}(\eta) \\ \quad + \eta \sum_{j=0}^{M_2-k-1} \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} \pi_{n,m} \hat{\tilde{P}}_{n,m;M_1-k,j}(\eta) & 0 \le k \le a-1 \\ \eta \sum_{i=0}^{M_1-a} \sum_{j=0}^{M_2-a} \sum_{n=0}^{M_1} \sum_{m=0}^{M_2} \pi_{n,m} \hat{\tilde{P}}_{n,m;i,j}(\eta) & k = a \\ 0 & \text{otherwise} \end{cases} \qquad (2.37)$$

4. A Simple Example to Illustrate the Proposed Technique

Consider one aircraft in operation. Suppose the aircraft is operational if it has both engine and avionics systems (one of each) installed and in operating condition. Suppose further that there are a total of 2 engines and 2 avionics systems in the organization. Both engines are assumed to be repaired by an engine technician that is not capable of repairing the avionics system, and can only work on one system at a time. Suppose that the organization has a single engine technician and 2 avionics technicians. Let $\lambda = 1$ aircraft per day be the total failure rate of the aircraft, the pilot reports failure due to engine with probability $p_1 = 0.4$, and due to avionics failure with probability $p_2 = 0.4$, and due to a failure of both systems with probability $p_{12} = 0.2$. Let $\mu_1 = 1$ engine per day be the repair rate of the engine, and $\mu_2 = 1$ avionics systems per day.

A graph of the state space and the transition rates for each state of the process $\{X_1(t), X_2(t);\ t \geq 0\}$ is shown in Figure (2.4).

Consider $\{X_1(t), X_2(t);\ t \geq 0\}$ as a Markov process on the state space $\{S_2, S_1, S_0\}$ as defined below:

$$S_2 = \{(2,0)\ (2,1)\ (2,2)\}$$

$$S_1 = \{(1,0)\ (1,1)\ (1,2)\}$$

$$S_0 = \{(0,0)\ (0,1)\ (0,2)\}$$

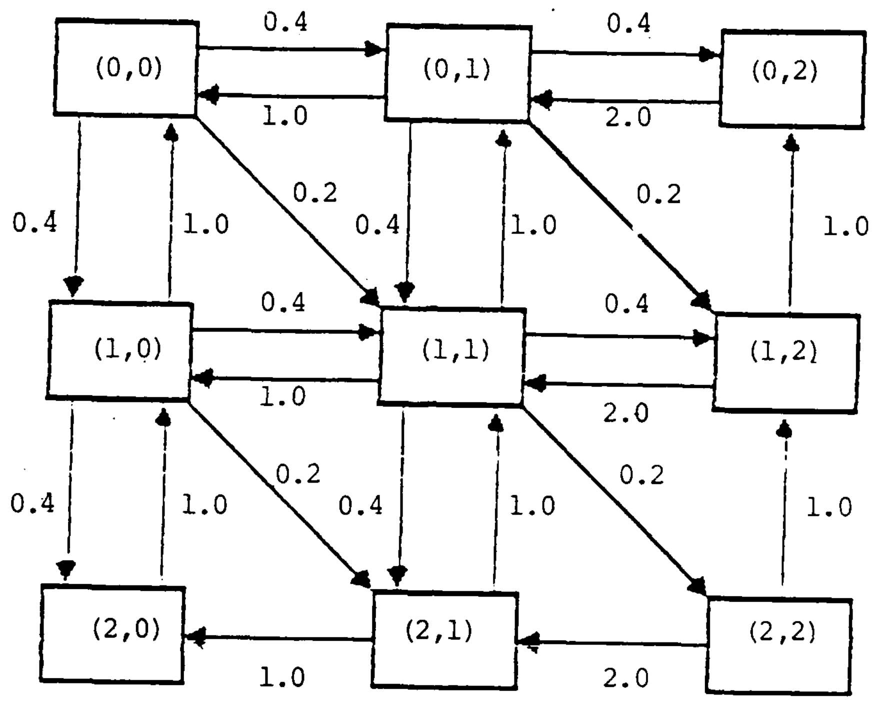

Figure 2.4. Transition Rates: A Simple Example to Illustrate the Matrix Geometric Approach

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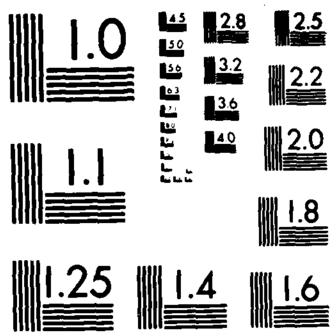

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Let:

$$\vec{P}_2(t) = \{P_{20}(t), P_{21}(t), P_{22}(t)\}$$

$$\vec{P}_1(t) = \{P_{10}(t), P_{11}(t), P_{12}(t)\}$$

$$\vec{P}_0(t) = \{P_{00}(t), P_{01}(t), P_{02}(t)\}$$

Define the failure and repair matrices $F^{(n)}$, $G_1^{(n)}$ and $G_2^{(n)}$ as follows:

$$F^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_1^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G_2^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad G_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$G_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let Q be the infinitesimal generator for the process. Therefore, using Proposition (2.1), we get:

$$Q = \begin{bmatrix} B_0 & C_0 & 0 \\ D_1 & B_1 & C_1 \\ 0 & D_2 & B_2 \end{bmatrix}$$

where:

$$B_0 = -(F^{(0)}+G_2^{(0)}) + G_2^{(0)}A^T + p_2F^{(0)}A = \begin{bmatrix} -1 & 0.4 & 0 \\ 1 & -2 & 0.4 \\ 0 & 2 & -2 \end{bmatrix}$$

$$B_1 = -(F^{(1)}+G_1^{(1)}+G_2^{(1)}) + G_2^{(1)}A^T + p_2F^{(1)}A = \begin{bmatrix} -2 & 0.4 & 0 \\ 1 & -3 & 0.4 \\ 0 & 2 & -3 \end{bmatrix}$$

$$B_2 = -(G_1^{(2)}+G_2^{(2)}) + G_2^{(2)}A^T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$C_0 = p_1 F^{(0)} + p_{12} F^{(0)} A = \begin{bmatrix} 0.4 & 0.2 & 0 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 = p_1 F^{(1)} + p_{12} F^{(1)} A = \begin{bmatrix} 0.4 & 0.2 & 0 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_1 = G_1^{(1)} \; ; \qquad D_2 = G_1^{(2)}$$

Hence, the infinitesimal generator, Q, becomes:

$$Q = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \end{array} \begin{array}{c} \begin{array}{ccccccccc} (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) \end{array} \\ \left[\begin{array}{ccccccccc} -1 & 0.4 & 0 & 0.4 & 0.2 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0.4 & 0 & 0.4 & 0.2 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0.4 & 0 & 0.4 & 0.2 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0.4 & 0 & 0.4 & 0.2 \\ 0 & 0 & 1 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -3 \end{array} \right] \end{array}$$

To compute the limiting probabilities, we need the following:

$$H_2 = B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix} ; \quad H_2^{-1} = \begin{bmatrix} -1 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 \\ -0.333 & -0.333 & -0.333 \end{bmatrix}$$

$$H_1 = B_1 + C_1(-H_2^{-1})D_2 = \begin{bmatrix} -1.5 & 0.5 & 0.0 \\ 1.2667 & -2.7333 & 0.4667 \\ 0 & 2 & -3 \end{bmatrix}$$

Therefore:

$$H_1^{-1} = \begin{bmatrix} -.8074 & -.1667 & -.0259 \\ -.4222 & -.5 & -.0778 \\ -.2815 & -.333 & -.3852 \end{bmatrix}$$

$$H_0 = B_0 + C_0(-H_1^{-1})D_1 = \begin{bmatrix} -.5926 & .5667 & .0259 \\ 1.2252 & -1.7333 & .5081 \\ 0 & 2 & -2 \end{bmatrix}$$

From Proposition (2.2), we get:

$$\vec{\pi}_0 C_0 = 0 , \quad \vec{\pi}_0 e = 1$$

Therefore:

$$\vec{\pi}_0 = [.6175 \quad .2987 \quad .0839]$$

$$\vec{\pi}_1 = \vec{\pi}_0 C_0(-H_1^{-1}) = [.3188 \quad .1826 \quad .0483]$$

To renormalize,

$$b = \vec{\pi}_0 e + \vec{\pi}_1 e = 1.5497$$

Therefore, dividing by b we obtain:

$$\vec{\pi}_0 = [.3985 \quad .1927 \quad .0541]$$

$$\vec{\pi}_1 = [.2057 \quad .1178 \quad .0312]$$

$$\vec{\pi}_2 = \vec{\pi}_1 C_1 (-H_2^{-1}) = [.1343 \quad .0520 \quad .0079]$$

To renormalize,

$$b = \sum_{n=0}^{2} \vec{\pi}_n e = 1.1941$$

Therefore, we obtain:

$$\vec{\pi}_0 = [.3337 \quad .1614 \quad .0453]$$

$$\vec{\pi}_1 = [.1723 \quad .0987 \quad .0261]$$

$$\vec{\pi}_2 = [.1125 \quad .0435 \quad .0066]$$

Therefore the readiness is:

$$P\{D(0) = 0\} = \sum_{i=0}^{2} \pi_{i,2} + \sum_{j=0}^{1} \pi_{2,j}$$

Hence,

$$P\{D(0) = 0\} = .234$$

and

$$P\{D(0) = 1\} = .766$$

Assume there is a warning that is exponentially distributed with mean = 1 day. The failure rate during this period drops from $\lambda = 1$ aircraft per day to $\lambda' = 0.5$ aircraft per day, and $\mu_1 = 1$, $\mu_2 = 1$ modules per day increase to $\mu_1' = 2$, $\mu_2' = 2$ modules per day. To find the readiness at the time of the attack, we need to determine the parameters of the new stochastic process $\{\tilde{X}_1(t), \tilde{X}_2(t);\ t \geq 0\}$. The rate parameters are state dependent as follows:

State (i,j)	$\tilde{f}(i,j)$	$\tilde{g}_1(i,j)$	$\tilde{g}_2(i,j)$
(0,0)	.5	0	0
(0,1)	.5	0	2
(0,2)	0	0	4
(1,0)	.5	2	0
(1,1)	.5	2	2
(1,2)	0	2	4
(2,0)	0	2	0
(2,1)	0	2	2
(2,2)	0	2	4

Hence, the matrices $\tilde{F}^{(n)}$, n = 0,1, $\tilde{G}_1^{(n)}$, n = 1,2, and $\tilde{G}_2^{(n)}$, n = 0,1,2 can be determined using the above rates.

$$\tilde{B}_0 = -(\tilde{F}^{(0)}+\tilde{G}_2^{(0)}) + \tilde{G}_2^{(0)}A^T + p_2\tilde{F}^{(0)}A = \begin{bmatrix} -.5 & .2 & 0 \\ 2 & -2.5 & .2 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\tilde{B}_1 = -(\tilde{F}^{(1)}+\tilde{G}_1^{(1)}+\tilde{G}_2^{(1)}) + \tilde{G}_2^{(1)}A^T + p_2\tilde{F}^{(1)}A = \begin{bmatrix} -2.5 & .2 & 0 \\ 2 & -4.5 & .2 \\ 0 & 4 & -6 \end{bmatrix}$$

$$\tilde{B}_2 = -(\tilde{G}_1^{(2)} + \tilde{G}_2^{(2)}) + \tilde{G}_2^{(2)}A^T = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

$$\tilde{C}_0 = p_1\tilde{F}^{(0)} + p_{12}\tilde{F}^{(0)}A = \begin{bmatrix} .2 & .1 & 0 \\ 0 & .2 & .1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{C}_1 = p_1\tilde{F}^{(1)} + p_{12}\tilde{F}^{(1)}A = \begin{bmatrix} .2 & .1 & 0 \\ 0 & .2 & .1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{D}_1 = \tilde{G}_1^{(1)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\tilde{D}_2 = \tilde{G}_1^{(1)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$E_2 = \tilde{B}_2 - \eta I = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 4 & -7 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} -.333 & 0 & 0 \\ -.1333 & -.2 & 0 \\ -.0762 & -.1143 & -.1429 \end{bmatrix}$$

$$E_1 = \tilde{B}_1 + \tilde{C}_1[-E_2^{-1}]\tilde{D}_2 - \eta I = \begin{bmatrix} -3.34 & .24 & 0 \\ 2.0686 & -5.3971 & .2286 \\ 0 & 4 & -7 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} -.3081 & -.014 & -.005 \\ -.121 & -.1954 & -.0064 \\ -.0692 & -.1117 & -.1465 \end{bmatrix}$$

$$E_0 = \tilde{B}_0 + \tilde{C}_0[-E_1^{-1}]\tilde{D}_1 - \eta I = \begin{bmatrix} -1.3526 & .2447 & .0015 \\ 2.0622 & -3.3995 & .2319 \\ 0 & 4 & -5 \end{bmatrix}$$

Using Proposition (2.4), we get the following values for $\hat{\tilde{P}}_{n,m;i,j}(n)$:

(i,j) / (n,m)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(0,0)	.8370	.0640	.0032	.0637	.0219	.0016	.0058	.0024	.0003
(0,1)	.5270	.3522	.0165	.0505	.0297	.0060	.0051	.0025	.0004
(0,2)	.4296	.2818	.2132	.0404	.0238	.0048	.0040	.0020	.0003
(1,0)	.5312	.0496	.0026	.3488	.0284	.0016	.0289	.0084	.0004
(1,1)	.4179	.1567	.0099	.1567	.2126	.0092	.0198	.0141	.0030
(1,2)	.3616	.1701	.0666	.1011	.1283	.1495	.0125	.0086	.0018
(2,0)	.3541	.0331	.0018	.2325	.0189	.0011	.3526	.0056	.0003
(2,1)	.3088	.0759	.0047	.1557	.0926	.0041	.1490	.2079	.0013
(2,2)	.2798	.0920	.0217	.1179	.0896	.0451	.0887	.1212	.1441

The following are the values for $\pi_{n,m}\hat{\tilde{P}}_{n,m;i,j}(\eta)$:

(i,j) / (n,m)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(0,0)	.2793	.0214	.0011	.0213	.0073	.0005	.0019	.0008	.0001
(0,1)	.0867	.0568	.0027	.0082	.0048	.0010	.0008	.0004	.0001
(0,2)	.0194	.0127	.0096	.0018	.0011	.0002	.0002	.0001	0
(1,0)	.0915	.0085	.0004	.0601	.0049	.0003	.0050	.0014	.0001
(1,1)	.0412	.0155	.0010	.0155	.0210	.0009	.0020	.0014	.0003
(1,2)	.0094	.0044	.0017	.0026	.0033	.0039	.0003	.0002	0
(2,0)	.0398	.0037	.0002	.0262	.0021	.0001	.0397	.0006	0
(2,1)	.0134	.0033	.0002	.0068	.0040	.0002	.0065	.0090	.0001
(2,2)	.0018	.0006	.0001	.0008	.0006	.0003	.0006	.0008	.0010

From equation (2.35) the expected probabilities for the number of modules down at the time of attack become:

State: (i,j):	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
$E[\tilde{P}_{i,j}(W)]$:	.5825	.1269	.0170	.1433	.0491	.0074	.0570	.0147	.0017

Hence, from equation (2.36), the readiness is given by:

$$P\{D(0) = 0\} = .0978$$

$$P\{D(0) = 1\} = .9022$$

The example was used only to show a simple application for the method of computing the operational readiness of the aircraft-squadron using the matrix geometric approach. The results obtained were compared with the results obtained by writing down the generator and solving for the limiting distribution, and for the Laplace transforms. The results obtained were the same as those obtained using the matrix geometric approach. The matrix geometric method has the attractiveness that at any time we are working with small-size matrices, which allows for solving a large-scale problem. As shown before, if we are considering a system with $M_1 = M_2 = 100$ modules, then the infinitesimal generator will be of size $10{,}201 \times 10{,}201$, which is too large to calculate the limiting distribution and Laplace transform by solving $\vec{\pi}Q = 0$, $\vec{\pi}\vec{e} = 1$ directly. Using the matrix geometric approach, we can only work with matrices of size 101×101, which are much smaller.

III. FORMULATION AND ANALYSIS OF A PROBLEM IN REPAIRMEN ALLOCATION FOR AN OPERATIONAL LOGISTICS SYSTEM

A. INTRODUCTION

This chapter models a repair shop maintenance process for a given aircraft squadron. The model formulation is based on a stochastic continuous-time Markovian decision process. Mathematical programming optimization techniques are implemented to determine the optimal repair policy. After this brief introduction, the objectives of this chapter will be presented, followed by the scope of work. The problem characteristics will be introduced, followed by the assumptions used in the model. The model is then developed, and sample solution of the model, using dynamic and linear programming techniques, are analyzed and verified. The optimization techniques are compared, and potential opportunities for improvements are suggested.

B. OBJECTIVE

The objective of this chapter is to formulate and analyze a repair policy for an aircraft squadron. Analysis of the repair policy means determining the various repair policies available to the shop supervisor, defining the effect of each policy on the squadron readiness, and selecting that repair policy which maximizes the expected number of aircraft ready to engage or "scramble" when an attack occurs.

C. SCOPE

An air-interceptor squadron is selected to represent the operational situation under study. The squadron is considered to operate independently of other squadrons. An aircraft requires a certain number of its systems to operate in order to be considered operational. The shop supervisor is assigned R repairmen capable of repairing all types of systems installed on the aircraft (i.e., general technicians).

This study concentrates on finding the optimal way of distributing the repairmen among different shops. Here the term "optimal" refers to the maximization of the readiness of the squadron. Ultimately the readiness of the squadron is assessed by its ability to engage into air combat when called upon with the maximum number of interceptors in an attempt to minimize the number of enemy bombers reaching their targets.

D. APPROACH

To achieve the above objectives, the repair process of the squadron during peace-time or non-combat periods has to be analyzed. The effect of the repairmen assignment decisions constitutes the restrictions for determining the optimal repair policy. When an aircraft lands, the entire repair shop complex experiences simultaneous demands for several different types of repair. It is assumed that an aircraft consists of several types of systems; for example, engine (propulsion), communications, airframe systems, etc. The system is referred to as a module. Let the number of modules of type i down for repair at time t be $\{X_i(t), t \geq 0\}$.

The vector whose components are $X_i(t)$ for all K types of modules installed on the aircraft, is taken to be the state of the system. Because the probabilistic nature of the system state changes, the process $\vec{X}(t) \equiv \{X_1(t),\ldots,X_K(t); t \geq 0\}$ represents a multivariate stochastic process. It will be assumed that the state of the system at time (t+s) given the history to time t is a function only of the state at t, so the process can be handled as a multivariate-continuous-time Markov process. If the supervisor is able to make a decision on the required number of repairmen to be assigned to every shop, every time the state of the system changes, then the problem can be formulated as a multivariate-continuous-time Markov decision process. Although it may be unrealistic to assume that the personnel are so rapidly interchangeable, this is the situation that is examined.

Consider a system, S, which at any point of time must be in exactly one of s possible states, labeled 1,2,...,s. Suppose S operates from time zero to time T, where $T \to \infty$. When S is in state i, an action γ is chosen from a finite set Γ_i of possible actions, and the return rate is $f(i,\gamma)$ that depends only on the current state and action taken.

The policy is assumed to be a stationary policy. A policy is said to be stationary if it is non randomized and the decision it chooses at time t depends only on the state of the process at time t (Ross, 1983). It follows that if a stationary policy is employed then the sequence of states

$\{X(t), t \geq 0\}$ forms a continuous-time Markov process with a transition probability,

$$P_{i,j}(t) = P_{i,j}(t, \gamma(i)) . \tag{3.1}$$

The evolution of the system from state to state described by $P_{i,j}(t,\gamma(i))$ depends on the infinitesimal generator whose elements are $q(j|i,\gamma(i))$, $j = 1,2,\ldots,M$, where $0 \leq q(j|i,\gamma(i)) < \infty$, $j \neq i$, is the rate of transition from state i to state j when decision $\gamma(i)$ is employed, and $q(i|i,\gamma(i)) = -\sum_{\substack{i=1 \\ j\neq i}}^{M} q(j|i,\gamma(i))$.

Therefore a collection of the row vectors $q(j|i,\gamma(i))$ for $i = 1,\ldots,M$ will form an infinitesimal generator that uniquely determines a continuous-time Markov process. That is, the probability is $q(j|i,\gamma(i))dt + o(dt)$ that the system will be in state j at time $t+dt$, $0 \leq dt < \delta$, $\delta > 0$, given that it is in state i at time t and action $\gamma(i) \in \Gamma_i$ is always used in the interval $[t,t+\delta)$ when the system is in state i.

The preceding statement can be generalized for the multi-variate-continuous-time Markov process. A state can be represented by a vector whose components are the states for the marginal-continuous-time Markov process. Let m be the cardinality of the set of all possible state vectors; then the system can be found in one of m possible states labeled $1,2,\ldots,m$ at any point of time. When the system is in state i an action $\vec{\gamma}(i)$ is chosen from a finite set Γ_i. The dimensionality of $\vec{\gamma}(i)$ need not be the same as the vector

representing state i. The dimensionality of $\vec{\gamma}(i)$ depends on the type of actions considered.

For example, if $\{(X_1(t),X_2(t),X_3(t),X_4(t)),t \geq 0\}$ represents the number of modules of types 1, 2, 3 and 4 down, respectively, and the action to be determined is the total number of repairmen to be assigned to the maintenance facility, i.e., the repairmen are assumed to be capable of repairing all types of modules, then $\vec{\gamma}(i)$ is a scalar. More realistically, suppose that the supervisor is assigned R repairmen, and the action to be taken is to distribute the R repairmen among the four shops. Then $\vec{\gamma}(i)$ is of dimension 3. The first component of $\vec{\gamma}(i)$ will be R_1, the number of repairmen to be assigned to Shop 1. The second component is R_2, the number of repairmen assigned to Shop 2, and the third is R_3, the number of repairmen assigned to Shop 3. R_4, the number of repairmen assigned to Shop 4, is the balance, $R_4 = R-(R_1+R_2+R_3)$ and need not be a component in the vector $\vec{\gamma}(i)$.

Therefore, if a stationary policy is employed then the sequence of states $\{\vec{X}(t),t \geq 0\}$ where $\vec{X}(t)$ is a vector, forms a multi-variate continuous-time Markov process with a transition probability $P_{i,j}(t) = P_{i,j}(t,\vec{\gamma}(i))$. As in the univariate case there will be a unique infinitesimal generator for every policy to be considered.

E. PROBLEM CHARACTERISTICS

The problem is characterized by these main aspects:

- Demand process.
- Repair process.
- Repairmen assignment decisions.

1. Demand Process

The demand process depends on factors that include number of aircraft in operation, number of pilots assigned, probability of failure of modules, possibility of a simultaneous failure, i.e., resulting in a simultaneous demand.

2. Repair Process

The factors affecting the repair process can be categorized as:

- Factors related to the maintenance organization.
- Factors related to the operational organization.
- Other factors.

Maintenance organizational factors include the number in various skill categories of repairmen, the location of different shops, rate of repair, availability of tools and test instruments, and the cannibalization policy.

Operational factors include number of aircraft assigned to the squadron, type of missions the aircraft are conducting, pilot skills and training, failure rate of the modules installed, and the sortie frequency.

Other factors include weather conditions, foreign object damage (FOD), manufacture or urgent maintenance bulletins, changes in type of threat, and number of spares available.

The two major factors affecting the repair policy are the failure rate and the repair rate of each type. The failure rate of each type of module depends on the number of aircraft, the failure rate of a module, the conditional probability of the module failing given an aircraft reported failed, and the total number of spare modules available.

The repair rate of each type is a function of the repair rate of a single repairman or team, the number of repairmen assigned, and the number of modules of the same type down at a given time.

These two major factors will be considered in analyzing the repair policy in this chapter.

3. Repairmen Assignment Decisions

The repairmen assignment decisions are the responsibility of the maintenance commander (supervisor). The decision concerning the number of repairmen to be assigned to each repair shop when the state of the system changes should follow a (nearly) optimal repairman assignment policy; such a policy is referred to in this study as the repair policy. The factors affecting the repair policy can be classified as:

- dependent upon the logistics system of the squadron, or
- dependent upon the repair process.

The logistics system factors include the availability of maintenance tools, documentation, preventive maintenance program, spare parts when needed, quality assurance program, and many others. In what follows the influence of such factors will be left implicit.

The repair process factors include repair time (diagnostic, removal of module from aircraft, ordering parts, repair of a failed module, testing of a module after repair, quality assurance inspection), and the queueing discipline used in the shop. For example,first-come-first-serve, or a priority system may be employed.

The major factors affecting the assignment decisions are the number of modules down requiring repair of each type, and the number of repairmen assigned to the maintenance organization. Therefore, the maintenance commander has to choose one decision among a finite set of possible decisions while considering the combat effectiveness of the squadron. The combat effectiveness of the squadron during peace time (between combat occasions) can be assumed to represent the readiness of the squadron at any point in time, which can be represented by the expected number of aircraft ready (or up) to engage in combat at the time the squadron must respond to an attack.

This chapter presents a mathematical model that will identify an optimal repair policy among finite feasible policies that are available for the commander to follow. As will be seen, the policy is idealized so as to allow calculations to be made. It does not recognize all reasonable constraints to which the repair system is subject. Nevertheless it, or its extensions, should provide some useful guidance for a decision maker.

F. PROBLEM FORMULATION

Consider an aircraft squadron that consists of a number, a, of failure-prone aircraft each susceptible to failure from one or both of two causes. Let λ denote the overall Markovian failure rate of an individual aircraft, and let P_1 denote the conditional probability that a failure requires just Type 1 (e.g., engine) repair, P_2 the conditional probability that the failure is of Type 2 (e.g., communications) and P_{12} the conditional probability that both Type-1 and Type-2 failures occur. Thus, $P_1 + P_2 + P_{12} = 1$, and the sequence of successive failure types is one of independently and identically distributed random variables. Suppose that there are M_1 and M_2 modules of Type 1 and Type 2 respectively, assigned to the squadron where $\min\{M_1, M_2\} \geq a$. There are R repairmen assigned to the squadron where each repairman is capable of repairing both types of modules.

Next, let $X_i(t)$, $i = 1,2$, denote the number of modules of Type i, $i = 1,2$, that are in or require repair at time t. Thus the number of operational aircraft is $\min\{a, M_1-X_1(t), M_2-X_2(t)\}$ where it is assumed that an aircraft requires both modules (one of each) to be considered operational, and this is the number of aircraft that are failure-prone; the others are awaiting one or both of the Type-1 and/or Type-2 modules. Finally, it is assumed that repair is Markovian (or exponential): μ_i denotes the rate at which an individual repair of Type i, $i = 1,2$, is completed at Shop i.

If Shop 1, where repair of Type 1 module is conducted, has R_1 repairmen assigned, where $0 \le R_1 \le R$, then the transition probabilities are, to terms of order dt:

<u>t</u>		<u>t+dt</u>	<u>Probability</u>	
(i,j)	$\rightarrow$	(i+1,j)	$P_1\ \lambda\ \min\{a, M_1-i, M_2-j\}dt$	
	$\rightarrow$	(i,j+1)	$P_2\ \lambda\ \min\{a, M_1-i, M_2-j\}dt$	
	$\rightarrow$	(i+1,j+1)	$P_{12}\lambda\ \min\{a, M_1-i, M_2-j\}dt$	(3.2)
	$\rightarrow$	(i-1,j)	$\mu_1\ \min\{R_1, i\}dt$	
	$\rightarrow$	(i,j-1)	$\mu_2\ \min\{R-R_1, j\}dt$	

The above assumptions and formulation specify the stochastic process $\{X_1(t), X_2(t), t \ge 0\}$ as a bivariate Markov birth-and-death process. The limiting probability exists and can be found by solving a system of linear equations.

G. DECISION MODEL FORMULATION: A REPAIR POLICY

A repair policy is a rule for making decisions after observing the system. The policies that are considered in this study are the type of policies that depend only upon the observed state of the system at time t, $\{X_1(t), X_2(t)\}$, and the possible decisions available, where decisions can only be taken when the state of the system changes. Therefore each policy can be completely characterized by a vector that prescribes the decision to be taken when the system is in state (i,j), $i = 0,1,\ldots,M_1$; $j = 0,1,\ldots,M_2$. The components of such a vector represent the number of repairmen to be assigned to Shop Type 1 when the system is in state (i,j).

All policies to be considered are stationary policies. That is, whenever the process is in state (i,j), the decision to be made is the same for all values of t.

The objective of the maintenance commander is to maximize the squadron readiness, which will be measured, as stated before, by the expected number of operational aircraft. Alternative measures are possible, and may even be more appropriate. To obtain a mathematical expression for the commander's objective, define the following:

$$P_{nm,ij}(t) = P\{X_1(t) = i, X_2(t) = j \mid X_1(0) = n, X_2(0) = m\} \quad (3.3)$$

Let $$P_{ij}(t) = P_{nm,ij}(t), \quad \text{then} \quad (3.4)$$

$$\pi_{ij} = \lim_{t\to\infty} P_{ij}(t) \quad (3.5)$$

i.e., $\pi_{i,j}$ is the limiting probability of observing the system to be in state (i,j).

All states of the Markov process defined by (3.2) communicate, so the Markov process is irreducible. Hence the limiting distribution exists and is independent of the initial condition. Since every state is a positive recurrent state, the limiting probabilities $\pi_{i,j}$ are all positive and form a probability distribution; (i.e., $\pi_{i,j} > 0$).

Let S be the collection of all possible states for the process $\{X_1(t), X_2(t); t \geq 0\}$.

$$S = \{(i,j): i = 0,1,\ldots,M_1; j = 0,1,\ldots,M_2\} .$$

The cardinality of the set S is $(M_1+1)(M_2+1)$. When the process is in state (i,j), there are $f(i,j) = \min\{a, M_1-i, M_2-j\}$ operational aircraft. Therefore the commander's objective stated mathematically is:

$$\max_{\text{all } \gamma(i,j)} \sum_{(i,j)\in S} f(i,j)\pi_{i,j}$$

where $\gamma(i,j)$ is the action taken whenever the process is in state (i,j).

To compute the optimal repair policy, two optimization approaches will be used, linear programming and dynamic programming.

To simplify the notation, it may be observed that a one-to-one mapping between the set $N = \{1,2,\ldots,(M_1+1)(M_2+1)\}$ and the set S can be defined. Figure (3.1) shows a correspondence between N and S. Let $n \in N$, and $(i,j) \in S$. We need to determine the values of i and j if n is known. We note that,

$$\text{if } n \leq M_2+1 \quad \text{then} \quad i = 0$$

$$\text{and if} \quad M_2+1 < n \leq 2(M_2+1) \quad \text{then} \quad i = 1$$

and if $2(M_2+1) < n \leq 3(M_2+1)$ then $i = 2$

Hence, if $n_1(M_2+1) < n \leq (n_1+1)(M_2+1)$ then $i = n_1$

The general form is

$$i(M_2+1) < n \leq (i+1)(M_2+1)$$

Therefore, if n is known, then i must satisfy:

$$\frac{n}{M_2+1} - 1 \leq i < \frac{n}{M_2+1}$$

Note that i must be integer. Therefore: if

$$\frac{n}{M_2+1} \text{ is integer, then } i = \frac{n}{M_2+1} - 1$$

Suppose that $\frac{n}{M_2+1}$ is not integer, then:

$$\left\lceil \frac{n}{M_2+1} - 1 \right\rceil \leq i \leq \left\lfloor \frac{n}{M_2+1} \right\rfloor$$

where $\lceil X \rceil$ is defined to the smallest integer $\geq$ X, and $\lfloor X \rfloor$ the largest integer $\leq$ X. But,

$$\left\lceil \frac{n}{M_2+1} - 1 \right\rceil = \left\lceil \frac{n}{M_2+1} - 1 \right\rceil = \left\lfloor \frac{n}{M_2+1} \right\rfloor$$

Therefore,

$$i = \left\lceil \frac{n}{M_2+1} - 1 \right\rceil \tag{3.6}$$

$$n = i(M_2+1) + j + 1 \qquad (3.7)$$

$$j = n-i(M_2+1) - 1 \qquad (3.8)$$

To illustrate the above, consider the case where $M_1 = 2$ and $M_2 = 3$. A graph of the state space is shown in Figure (3.1), where each state is numbered by n and the corresponding (i,j) values. Table (3.1) shows the correspondence between n and (i,j) computed using equations (3.6) and (3.8). Examination of the table shows clearly the usefulness of equations (3.6) and (3.8).

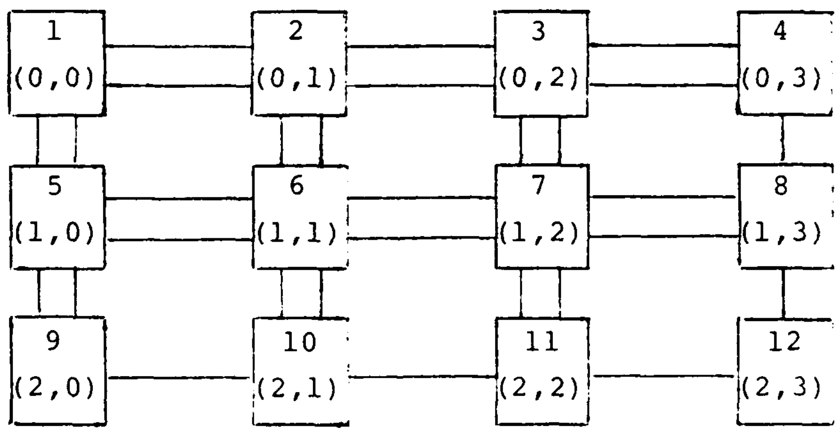

Figure 3.1. A Correspondence Between n and (i,j)

To generalize the above mapping, we derive equations similar to equations (3.6) and (3.8) for the case of a trivariate-continuous-time Markov process.

Consider a Markov process $X(t) = \{X_1(t),X_2(t),X_3(t);t \geq 0\}$ operating on a state space $S = \{(i,j,k): i = 0,\ldots,M_1,$

TABLE 3.1

THE CORRESPONDENCE BETWEEN N AND (I,J) USING EQUATIONS (3.6) AND (3.8)

n	$i = \left\lceil \frac{n}{M_2+1} - 1 \right\rceil$	$j = n - i(M_2+1) - 1$
1	0	0
2	0	1
3	0	2
4	0	3
5	1	0
6	1	1
7	1	2
8	1	3
9	2	0
10	2	1
11	2	2
12	2	3

$j = 0,\ldots,M_2; k = 0,\ldots,M_3\}$. Let $N = \{1,2,\ldots,(M_1+1)(M_2+1)(M_3+1)\}$. Let $n \in N$, then we need to determine (i,j,k).

If $\quad n \le (M_2+1)(M_3+1) \quad$ then $\quad i = 0$

If $\quad (M_2+1)(M_3+1) < n \le 2(M_2+1)(M_3+1) \quad$ then $\quad i = 1$

If $\quad n_1(M_2+1)(M_3+1) < n \le (n_1+1)(M_2+1)(M_3+1) \quad$ then $\quad i = n_1$

Therefore, if n is known, then i must satisfy:

$$\frac{n}{(M_2+1)(M_3+1)} - 1 \leq i < \frac{n}{(M_2+1)(M_3+1)}$$

Using the same argument used for the bivariate case we find that:

$$i = \left\lceil \frac{n}{(M_2+1)(M_3+1)} - 1 \right\rceil$$

Now, knowing both i and n, we need to determine j.

If $i = 0$, then:

If $n \leq (M_3+1)$ then $j = 0$

If $(M_3+1) < n \leq 2(M_3+1)$ then $j = 1$

If $n_2(M_3+1) < n \leq (n_2+1)(M_3+1)$ then $j = n_2$

Hence: If $i = 0$ then $j = \left\lceil \frac{n}{(M_3+1)} - 1 \right\rceil$

If $i = 1$, then:

If $(M_2+1)(M_3+1) < n \leq [(M_2+1)+1](M_3+1)$ then $j = 0$

If $[(M_2+1)+n_2](M_3+1) < n \leq [(M_2+1)+n_2+1](M_3+1)$ then $j = n_2$

Hence: If $i = 1$, then $j = \left\lceil \frac{n}{M_3+1} - (M_2+1) - 1 \right\rceil$

Therefore:

$$j = \left\lceil \frac{n}{M_3+1} - i(M_2+1) - 1 \right\rceil$$

$$n = i(M_2+1)(M_3+1) + j(M_3+1) + k + 1$$

Thus, the trivariate Markov process X(t), operating on the state space S, can be transformed into the univariate Markov process $\{Y(t), t \geq 0\}$ operating on the state space N by using the following equations:

$$n = i(M_2+1)(M_3+1) + j(M_3+1) + k + 1$$

$$i = \left\lceil \frac{n}{(M_2+1)(M_3+1)} - 1 \right\rceil$$

$$j = \left\lceil \frac{n}{M_3+1} - i(M_2+1) - 1 \right\rceil$$

$$k = n - i(M_2+1)(M_3+1) - j(M_3+1) - 1$$

Various trial runs have been conducted on the above equations for both bivariate and trivariate Markov processes, and found to be correct. This shows that a generalization for a K^{th} variate-Markov process is possible, as shown below:

Consider a K^{th}-variate-continuous-time-Markov process $X(t) \equiv \{X_1(t), X_2(t), \ldots, X_K(t); t \geq 0\}$ operating on the state space $S = \{(i_1, i_2, \ldots, i_k); i_j = 0, \ldots, M_j, j = 1, \ldots, K\}$. The cardinality of the state space is $\prod_{j=1}^{K} (M_j+1)$.

Let $N = \{1, 2, \ldots, \prod_{j=1}^{K} (M_j+1)\}$. Let $n \in N$. Then:

$$n = \sum_{j=1}^{K-1} i_j \prod_{\ell=j+1}^{K} (M_\ell+1) + i_K + 1$$

$$i_1 = \left\lceil \frac{n}{\prod_{j=2}^{K} (M_j+1)} - 1 \right\rceil$$

$$i_m = \left\lceil \frac{n}{\prod_{j=m+1}^{K} (M_j+1)} - \sum_{\ell=1}^{m-1} i_\ell \prod_{j=\ell+1}^{m} (M_j+1) - 1 \right\rceil \qquad (3.9)$$

$$2 \leq m \leq K-1$$

$$i_K = n - \sum_{j=1}^{K-1} i_j \prod_{\ell=j+1}^{K} (M_\ell+1) - 1$$

Considering different values for K, we get the following equations:

For K = 2:

$$n = i_1(M_2+1) + i_2 + 1$$

$$i_1 = \left\lceil \frac{n}{(M_2+1)} - 1 \right\rceil$$

$$i_2 = n - i_1(M_2+1) - 1$$

For K = 3:

$$n = i_1(M_2+1)(M_3+1) + i_2(M_3+1) + i_3 + 1$$

$$i_1 = \frac{n}{(M_2+1)(M_3+1)} - 1$$

$$i_2 = \frac{n}{(M_3+1)} - i_1(M_2+1) - 1$$

$$i_3 = n - i_1(M_2+1)(M_3+1) - i_2(M_3+1) - 1$$

For K = 4:

$$n = i_1(M_2+1)(M_3+1)(M_4+1) + i_2(M_3+1)(M_4+1) + i_3(M_4+1) + i_4 + 1$$

$$i_1 = \left\lceil \frac{n}{(M_2+1)(M_3+1)(M_4+1)} - 1 \right\rceil$$

$$i_2 = \left\lceil \frac{n}{(M_3+1)(M_4+1)} - i_1(M_2+1) - 1 \right\rceil$$

$$i_3 = \left\lceil \frac{n}{(M_4+1)} - i_1(M_2+1)(M_3+1) - i_2(M_3+1) - 1 \right\rceil$$

$$i_4 = n - i_1(M_2+1)(M_3+1)(M_4+1) - i_2(M_3+1)(M_4+1)$$
$$- i_3(M_4+1) - 1$$

For K = 5:

$$n = i_1(M_2+1)(M_3+1)(M_4+1)(M_5+1) + i_2(M_3+1)(M_4+1)(M_5+1)$$
$$+ i_3(M_4+1)(M_5+1) + i_4(M_5+1) + i_5 + 1$$

$$i_1 = \left\lceil \frac{n}{(M_2+1)(M_3+1)(M_4+1)(M_5+1)} - 1 \right\rceil$$

$$i_2 = \left\lceil \frac{n}{(M_3+1)(M_4+1)(M_5+1)} - i_1(M_2+1) - 1 \right\rceil$$

$$i_3 = \left\lceil \frac{n}{(M_4+1)(M_5+1)} - i_1(M_2+1)(M_3+1) - i_2(M_3+1) - 1 \right\rceil$$

$$i_4 = \left\lceil \frac{n}{(M_5+1)} - i_1(M_2+1)(M_3+1)(M_4+1) - i_2(M_3+1)(M_4+1) - i_3(M_4+1) - 1 \right\rceil$$

$$i_5 = n - i_1(M_2+1)(M_3+1)(M_4+1)(M_5+1) - i_2(M_3+1)(M_4+1)(M_5+1)$$
$$- i_3(M_4+1)(M_5+1) - i_4(M_5+1) - 1$$

Various examples carried out for the various cases and the above equations were found to give the correct correspondence between the set N and the corresponding state space S.

Therefore, it could be concluded that the set of equations (3.9) allow us to transform a K-dimensional-continuous-time Markov process operating on a finite discrete state space to a univariate-continuous-time Markov process operating on the finite state space $N = \{1,2,\ldots,\prod_{j=1}^{K}(M_j+1)\}$. This allows the analyst to carry out all necessary analysis on the univariate-continuous-time-Markov process, using all existing theorems and algorithms for the univariate process, and then retransforming the results back to the K-dimensional process using again the set of equations (3.9).

As shown above, in the case of bivariate Markov process, knowing n, the corresponding state (i,j) can be determined and vice versa, i.e., if (i,j) is known, then n can be determined. With such observations the bivariate continuous-time-Markov-process can be transformed into a univariate-continuous-time-Markov-process. Define the stochastic process $\{Y(t), t \geq 0\}$ to be a continuous-time-Markov process

operating on the state space $\{1,2,\ldots,(M_1+1)(M_2+1)\}$. Hence, if $Y(t) = n$ then $X_1(t) = i$, $X_2(t) = j$ where i,j are given by equations (3.6) and (3.8) respectively.

Let $P_{m,n}(t) = P\{Y(t) = n | Y(0) = m\}$

$$P_{m,n}(t) = P\{X_1(t) = i, X_2(t) = j | X_1(0) = i_o, X_2(0) = j_o\}$$

where:

$$i_o = \left\lceil \frac{m}{M_2+1} - 1 \right\rceil$$

$$j_o = m - i_o(M_2+1) - 1$$

i,j are given by Equations (3.6) and (3.8).

Let $\pi_n = \lim_{t\to\infty} P_{m,n}(t)$

$$= \lim_{t\to\infty} P\{X_1(t) = i, X_2(t) = j | X_1(0) = i_o, X_2(0) = j_o\}.$$

Using equations (3.3) to (3.5), the following was obtained:

$$\pi_n = \pi_{i,j}$$

Since $\pi_{i,j}$ was shown to exist and is positive, therefore it was concluded that π_n will exist and is strictly positive (i.e., $0 < \pi_n < 1$).

Let:

λ_n be the total failure rate when the process is in state n.

μ_{1n} be the total repair rate when the process is in state n for Shop 1.

μ_{2n} be the total repair rate when the process is in state n for Shop 2.

Therefore:

$$\lambda_n = \lambda\ f(n)$$

$$\mu_{1n} = \mu_1 \min(\gamma(n), i)$$

$$\mu_{2n} = \mu_2 \min(R-\gamma(n), j)$$

where:

$$f(n) = \min\{a, M_1-i, M_2-j\}$$

i and j are given by equations (3.6) and (3.8).

Figure (3.2) shows the corresponding transition graph for the process $\{Y(t), t \geq 0\}$. The transition for any state (i,j) is shown in Figure (3.3) for any repair policy.

H. LINEAR PROGRAMMING APPROACH

Let $\pi_{i,j}(\gamma)$ be the unconditional limiting probability that the system is in state (i,j) and decision γ is made, that is:

$$\pi_{i,j}(\gamma) = P\{\text{state} = (i,j) \text{ and decision} = \gamma\}\ ,$$

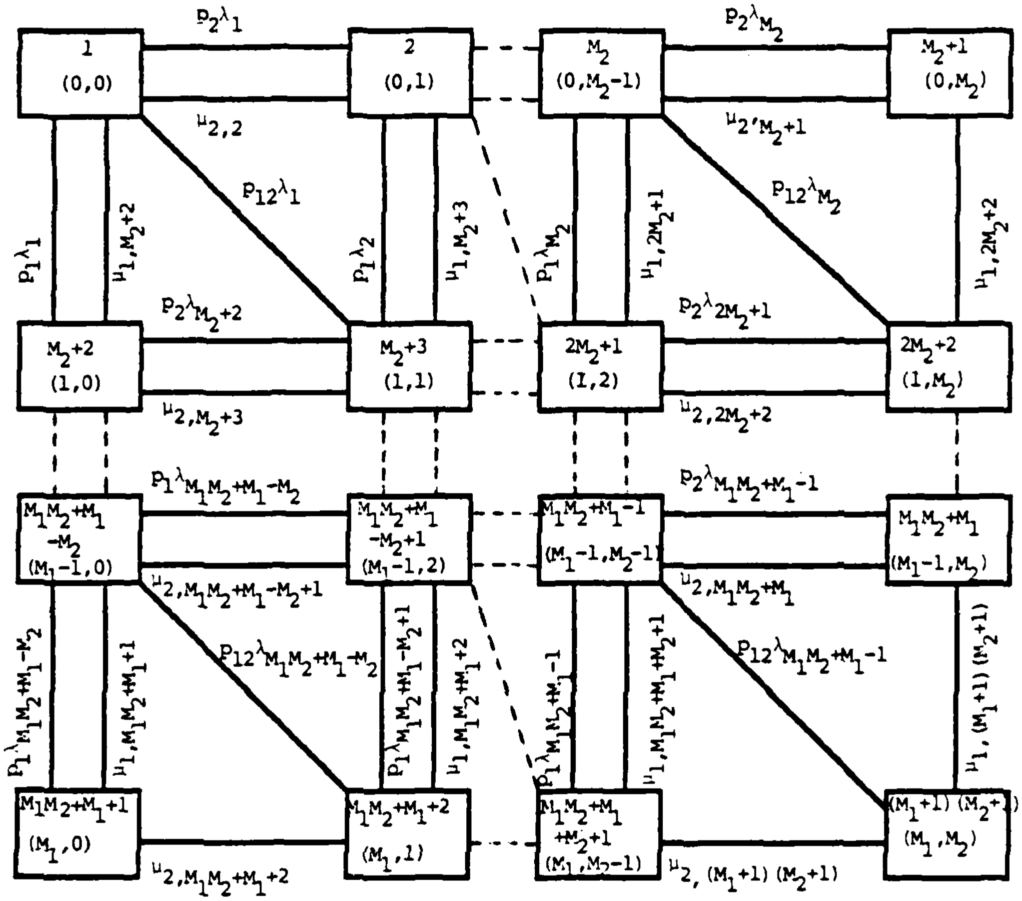

Figure 3.2. Transition Flow for the Failure-Repair Process

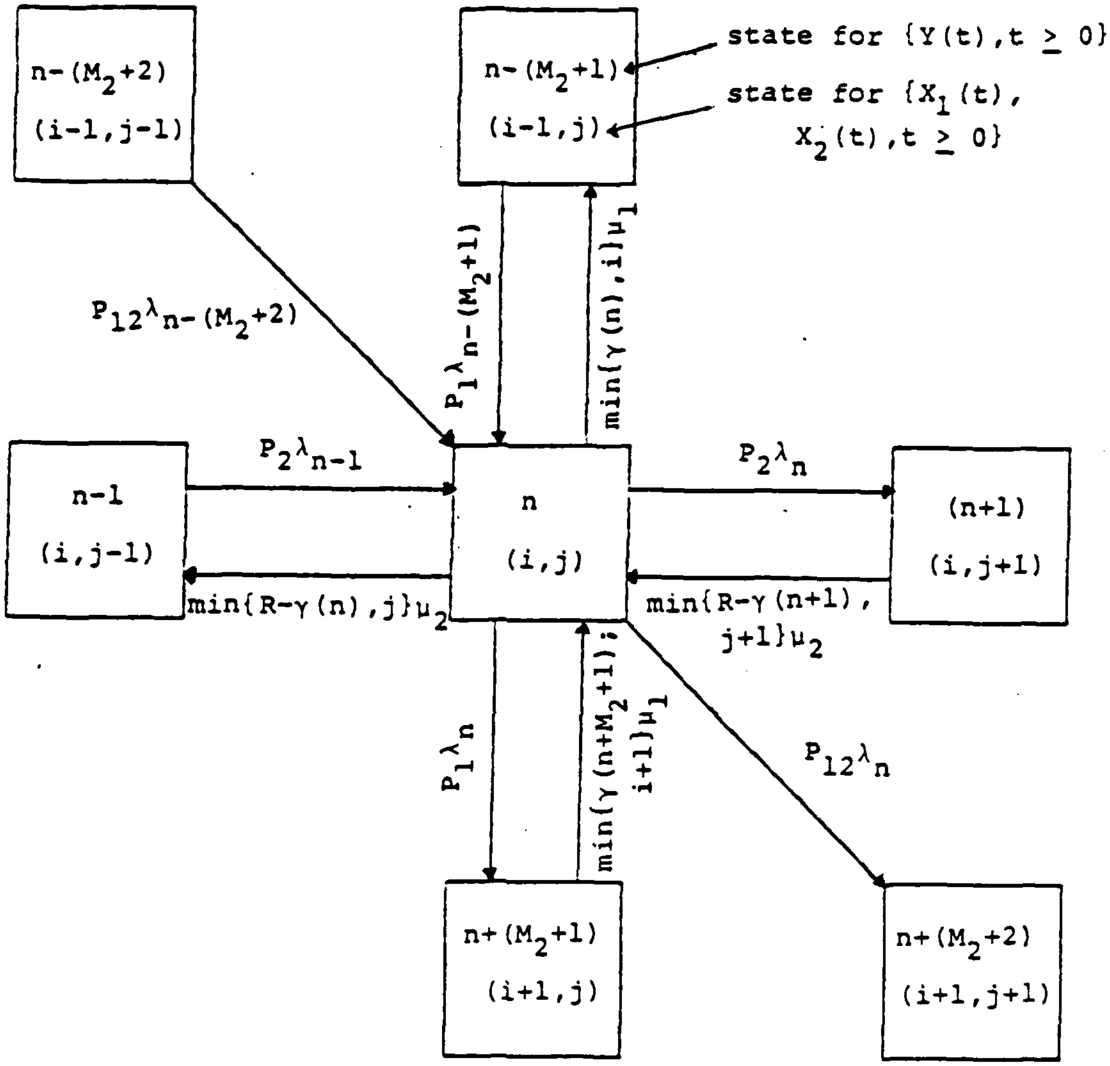

Figure 3.3. The Transitions Flow for State (i,j) for any Fepair Policy

Therefore,

$$\pi_n(\gamma) = \pi_{i,j}(\gamma) \tag{3.10}$$

where:

$$\pi_n(\gamma) = P\{\text{state} = n \text{ and decision} = \gamma\} .$$

These limiting probabilities were obtained from equations (3.6) and (3.8).

From the law of total probability, we get:

$$P\{\text{state} = (i,j)\} = \sum_{\gamma} P\{\text{state} = (i,j) \text{ and decision} = \gamma\}$$

$$\pi_n = \sum_{\gamma} \pi_n(\gamma) \tag{3.11}$$

Since π_n is a limiting distribution, then π_n is the unique solution for the following system of linear equations:

1. $$\sum_{n=1}^{(M_1+1)(M_2+1)} \pi_n = 1$$

2. $$\sum_{n=1}^{(M_1+1)(M_2+1)} \pi_n q(n,m) = 0 \qquad m = 1,2,\ldots,(M_1+1)(M_2+1)$$
(the forward equations)

and

3. $$\pi_n \geq 0 \qquad n = 1,2,\ldots,(M_1+1)(M_2+1) .$$

where: $q(n,m)$ is the $(n,m)^{th}$ element of the matrix Q, the infinitesimal generator of the Markov Process; i.e., $q(n,m)$ is thought of as the rate from state n to state m. Therefore $\pi_n(\gamma)$ must satisfy the following linear relations:

1. $$\sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} \pi_n(\gamma) = 1$$

2. $$\sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} \pi_n(\gamma) q_\gamma(n,m) = 0 \quad m = 1,2,\ldots,(M_1+1)(M_2+1)$$

and

3. $$\pi_n(\gamma) \geq 0 \quad \gamma = 0,1,2,\ldots,R \quad n = 1,0,2,\ldots,(M_1+1)(M_2+1).$$

The long run expected number of aircraft that are operational is given by:

$$\text{E[Number of aircraft operational]} = \sum_{n=1}^{(M_1+1)(M_2+1)} f(n)\pi_n$$

$$= \sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} f(n)\pi_n(\gamma)$$

Therefore the objective function is given by:

$$\text{maximize} \quad \sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} f(n)\pi_n(\gamma) \ .$$

$\pi_n^*(\gamma)$, $\gamma = 0,1,\ldots,R$, $n = 1,2,\ldots,(M_1+1)(M_2+1)$, that maximizes the objective function and satisfies the constraints is the solution for the following linear programming model:

<u>LP</u>

$$\max_{\pi} \sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} f(n)\pi_n(\gamma)$$

subject to:

$$\text{(i)} \quad \sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} \pi_n(\gamma) = 1$$

$$\text{(ii)} \quad \sum_{n=1}^{(M_1+1)(M_2+1)} \sum_{\gamma=0}^{R} \pi_n(\gamma)q_\gamma(n,m) = 0 \qquad m = 1,2,\ldots,(M_1+1)(M_2+1)$$

$$\text{(iii)} \quad \pi_n(\gamma) \geq 0 \qquad \gamma = 0,1,2,\ldots,R; \quad n = 1,2,\ldots,(M_1+1)(M_2+1)$$

Since the objective function is a linear function of the decision variable $\pi_n(\gamma)$; and the constraints form a set of linear equations, the above problem is a linear programming problem. Note that, even though $f(n)$ ($f(n) = \min\{a,M_1-i,M_2-j\}$) is a nonlinear function of i and j, it is not a function of $\pi_n(\gamma)$.

The value of $\pi_n(\gamma)$ will dictate the value of γ, where if $\pi_n(\gamma) > 0$ then the decision to be taken when in state n is

γ. This observation (Ross, 1983), allows us to solve the linear program, and the solution will always have γ as either 0 or 1.

The above formulated linear programming problem has $(M_1+1)(M_2+1)+1 = M_1M_2 + M_1 + M_2 + 2$ functional constraints. Since the infinitesimal generator Q is of rank $M_1M_2+M_1+M_2+1$, there is a constraint that is linearly dependent on the other constraints. Hence the problem has a single redundant constraint, and the number of constraints in the L-P can be reduced at no loss by dropping one of the constraints (ii) in the L-P.

The L-P can be solved by the simplex method, and its solution has some interesting properties (Ross, 1983). The solution will contain $(M_1+1)(M_2+1) = M_1M_2+M_1+M_2+1$ basic variables $\pi_n(\gamma) \geq 0$. Since the repair policy was assumed to be stationary, the policy is <u>deterministic</u>, meaning that one and only one decision will be chosen each time the process enters any given state. It follows that only one $\pi_n(\gamma^*) > 0$, while the $\pi_n(\gamma) = 0$ for $\gamma = 0,1,\ldots,R$, $\gamma \neq \gamma^*$. Since the Markov process is irreducible and since every state is positive recurrent, it follows that $\pi_n(\gamma^*) > 0$ for $n = 1,2,\ldots,(M_1+1)(M_2+1)$. Therefore, $\pi_n(\gamma) > 0$ for exactly one $\gamma = 0,1,\ldots,R$, for each $n = 1,2,\ldots,(M_1+1)(M_2+1)$.

The L-P has $(R+1)(M_1+1)(M_2+1)$ decision variables. Suppose that the problem at hand has $M_1 = 50$, $M_2 = 60$, $R = 20$. Then:

Number of decision variables = (21)(51)(61) = 65,331 variables.

Number of constraints = 3112 constraints.

Therefore the 'practical' problems tend to be large under this formulation.

I. DYNAMIC PROGRAMMING APPROACH

The finite-state-continuous-time-Markov process being analyzed has been shown to have a limiting distribution. Therefore, the <u>policy-iteration method</u> for the solution of the sequential decision process, developed by R. Howard (Howard, 1969), can be used.

Howard states that every completely ergodic Markov process with rewards will have a gain, g, given by:

$$g = \sum_{n=1}^{N} \pi_n h(n) \tag{3.12}$$

where:

N = number of states

π_n = limiting probability of being in state n

$h(n)$ = expected immediate return in state n defined by:

$$h(n) = r(n,n) + \sum_{m \neq n} q(n,m) r(n,m) \qquad n = 1,2,\ldots,N \tag{3.13}$$

where:

$q(n,m)$ = rate of transition from state n to state m.

$r(n,m)$ = reward associated with the transition from n to m.

Since there is no reward gained by transitioning from state n to state m, $r(n,m) = 0$ for $n \neq m$, and $h(n) = r(n,n)$. Hence, for the repair scheduling problem:

$$g = \sum_{n=1}^{(M_1+1)(M_2+1)} \pi_n f(n) \tag{3.14}$$

where:

$$f(n) = \min\left\{a, M_1 - \left\lceil \frac{n}{M_2+1} - 1 \right\rceil, M_2 - n + \left\lceil \frac{n}{M_2+1} - 1 \right\rceil (M_2+1)+1\right\}$$

Since at each state a decision has to be chosen from a number of decisions, therefore it can be thought of as having several Markov processes, and the objective is to know which one yields the highest expected number of aircraft operational, in the long run.

It is possible to describe the policy by a decision vector Γ whose elements represent the number of the decision selected in each state; i.e., the i^{th} element of the vector Γ indicates the number of repairmen to be assigned to Shop 1 whenever the process is in state i.

An optimal policy is defined to be the policy that maximizes the gain, or average return per transition.

The optimal policy can be found using the policy-iteration method which consists of the value-determination operation and the policy-improvement operation. Both operations can be used recursively as shown in Figure (3.4), until the optimal policy is found.

Let:

$V_n(t)$ = Total expected reward the system will earn in a time t if it starts in state n.

$V_n(t+dt)$ can be related to $V_n(t)$ by considering the following transitions and rewards.

<u>t</u>	<u>t+dt</u>	<u>reward</u>	<u>probability</u>
n	n	$f(n)dt+V_n(t)$	$1 - \sum_{n\neq m} q(n,m)dt$
	$(m\neq n)$	$V_m(t)$	$q(n,m)dt \quad \forall m\neq n$

Therefore,

$$V_n(t+dt) = (1 - \sum_{n\neq m} q(n,m)dt)[f(n)dt + V_n(t)]$$

$$+ \sum_{n\neq m} q(n,m)dtV_m(t) + o(dt)$$

But,

$$q(n,n) = - \sum_{n\neq m} q(n,m)$$

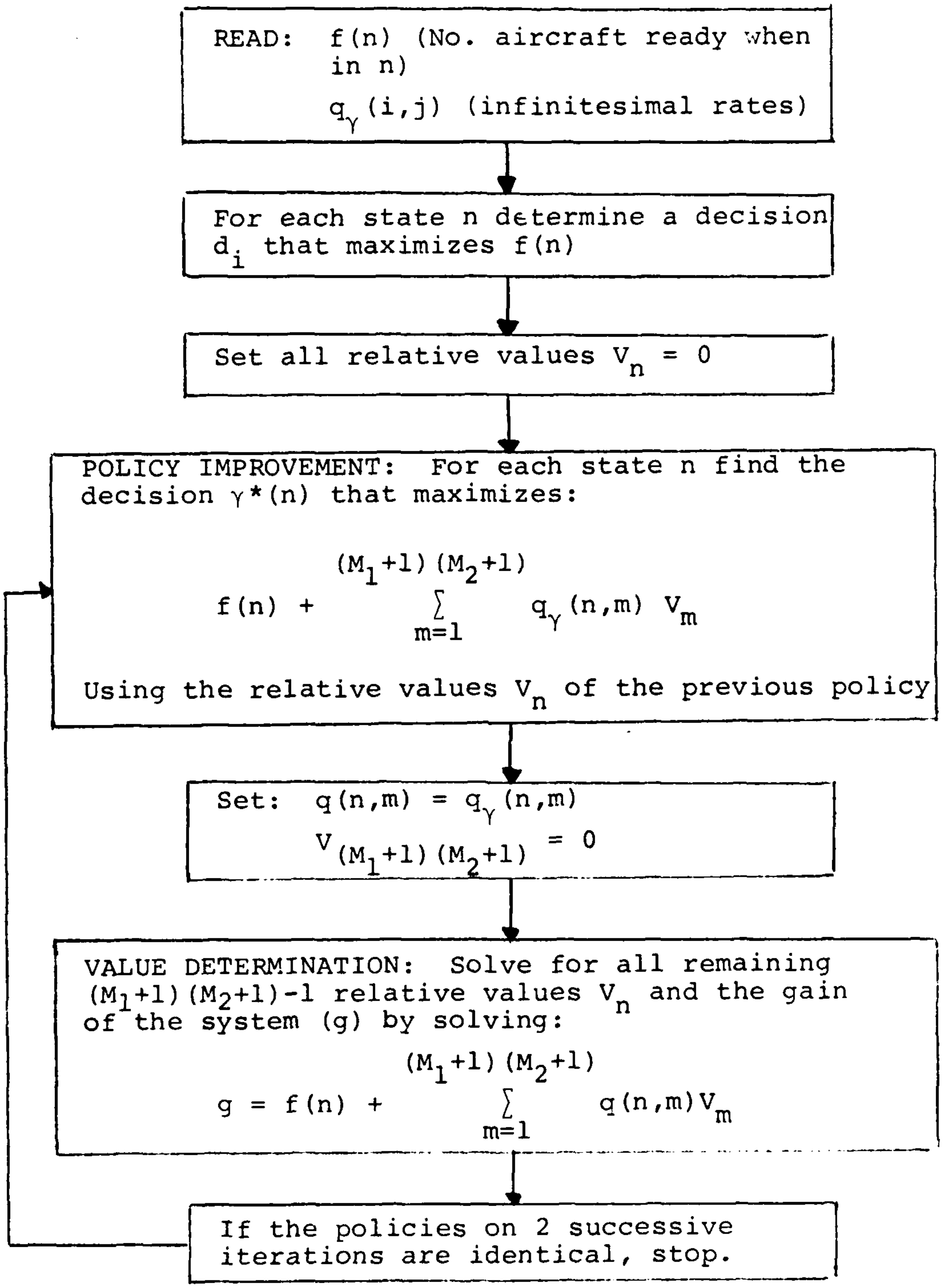

Figure 3.4. Iteration Cycle Schematic (Howard, 1969)

$$V_n(t+dt) = (1+q(n,n)dt)[f(n)dt+V_n(t)]$$

$$+ \sum_{n\neq m} q(n,m)dtV_m(t) + o(dt)$$

Subtracting $V_n(t)$ from both sides and dividing by dt, the above equation becomes:

$$\frac{V_n(t+dt)-V_n(t)}{dt} = f(n) + q(n,n)f(n)dt + q(n,n)V_n(t)$$

$$+ \sum_{n\neq m} q(n,m)V_m(t)$$

Let $dt \to 0$; the above equation becomes:

$$\frac{d}{dt} V_n(t) = f(n) + \sum_{m=1}^{(M_1+1)(M_2+1)} q(n,m)V_m(t) \tag{3.15}$$

$$n = 1,\ldots,(M_1+1)(M_2+1)$$

Changing to vector notation, this becomes:

$$\frac{d}{dt} \vec{V}(t) = \vec{F} + Q\,\vec{V}(t) \tag{3.16}$$

where:

$$\vec{V}^T(t) = [v_1(t),\ldots,v_{(M_1+1)(M_2+1)}(t)]$$

$$\vec{F}^T = [f(1),\ldots,f(M_1+1)(M_2+1)]$$

Q = infinitesimal generator.

The set of equations (3.16) is a matrix representation of equations (3.15). The equations are a set of linear, constant-coefficient differential equations that relate the total reward in time t from a start in state n to the quantities f(n) and q(n,m). But for large t (Howard, 1969)

$$V_n(t) = tg_n + V_n \qquad n = 1,\ldots,(M_1+1)(M_2+1)$$

Therefore:

$$g_n = \frac{dV_n(t)}{dt}$$

Using equation (3.15) we obtain:

$$g_n = f(n) + \sum_{m=1}^{(M_1+1)(M_2+1)} q(n,m)(tg_m+V_m)$$

$$g_n = f(n) + t \sum_{m=1}^{(M_1+1)(M_2+1)} g(n,m)g_m + \sum_{m=1}^{(M_1+1)(M_2+1)} g(n,m)V_m \tag{3.17}$$

For equation (3.17) to hold for all large t, we must have:

$$\sum_{m=1}^{(M_1+1)(M_2+1)} q(n,m)g_m = 0 \tag{3.18}$$

and

$$g(n) = f(n) + \sum_{m=1}^{(M_1+1)(M_2+1)} q(n,m)V_m \qquad (3.19)$$

The set of equations (3.19) consists of $(M_1+1)(M_2+1)$ equations with $(M_1+1)(M_2+1)+1$ unknowns ($(M_1+1)(M_2+1)V_m$'s and 1 g). Therefore set $V_{(M_1+1)(M_2+1)} = 0$ and solve for the rest.

Thus, for a given policy we can find the gain and the relative values of that policy by solving the $(M_1+1)(M_2+1)$ linear equations simultaneously with $V_{(M_1+1)(M_2+1)} = 0$.

The Policy-Improvement routine is then used to find a policy that has higher gain than the original policy.

$$\underset{\gamma}{\text{Max}}\ f(n) + \sum_{m=1}^{(M_1+1)(M_2+1)} q_\gamma(n,m)V_m$$

The optimal γ will be the decision to be taken when in state n; hence, a new policy has been developed. The procedure is then repeated until g is maximized, i.e., when the policies on 2 successive iterations are identical.

J. MODEL ILLUSTRATION

The following basic data were used as an example:

- An aircraft requires 2 types of modules (e.g., engines and communications) to be operational. Modules of Type 1 are engines and modules of Type 2 are communications equipment.
- The squadron is assigned exactly 2 aircraft.

- The total number of Type 1 and Type 2 modules available in the squadron are 2 and 3, respectively (i.e., $M_1 = 2$, $M_2 = 3$)

- The maintenance commander is assigned 3 repairmen, and every repairman is capable of repairing both modules.

- An aircraft is reported down with a rate $\lambda = 1$ aircraft per day.

- The probabilities of an aircraft down due to module failures of Type 1, Type 2 or both Types are $P_1 = .4$, $P_2 = .5$, $P_{12} = .1$ respectively.

- Modules of Type 1 and Type 2 are repaired with rates $\mu_1 = .5$ and $\mu_2 = 1$ modules per day.

A FORTRAN program has been written to compute the infinitesimal generator matrices, and create the output format suitable for optimality analysis. The linear programming was implemented using the LINDO interactive package. A FORTRAN program to implement the dynamic programming approach was written using the policy improvement technique. Appendix C presents the computer listings for the programs.

The graph of the above system is shown in Figure (3.5) where the repair rates are functions of the decision. The infinitesimal generators Q_γ for $\gamma = 0,1,2,3$ are illustrated in Figures (3.6), (3.7), (3.8) and (3.9) respectively. The infinitesimal generators Q for any policy can now be calculated using the infinitesimal generators Q_γ, $\gamma = 0,1,2,3$. This is done by picking the necessary rows from the four Q_γ matrices. For example, suppose the infinitesimal generator for the policy

$$\Gamma = (1\ 2\ 2\ 0\ 3\ 2\ 1\ 0\ 2\ 2\ 2\ 2) = (\gamma(n))$$

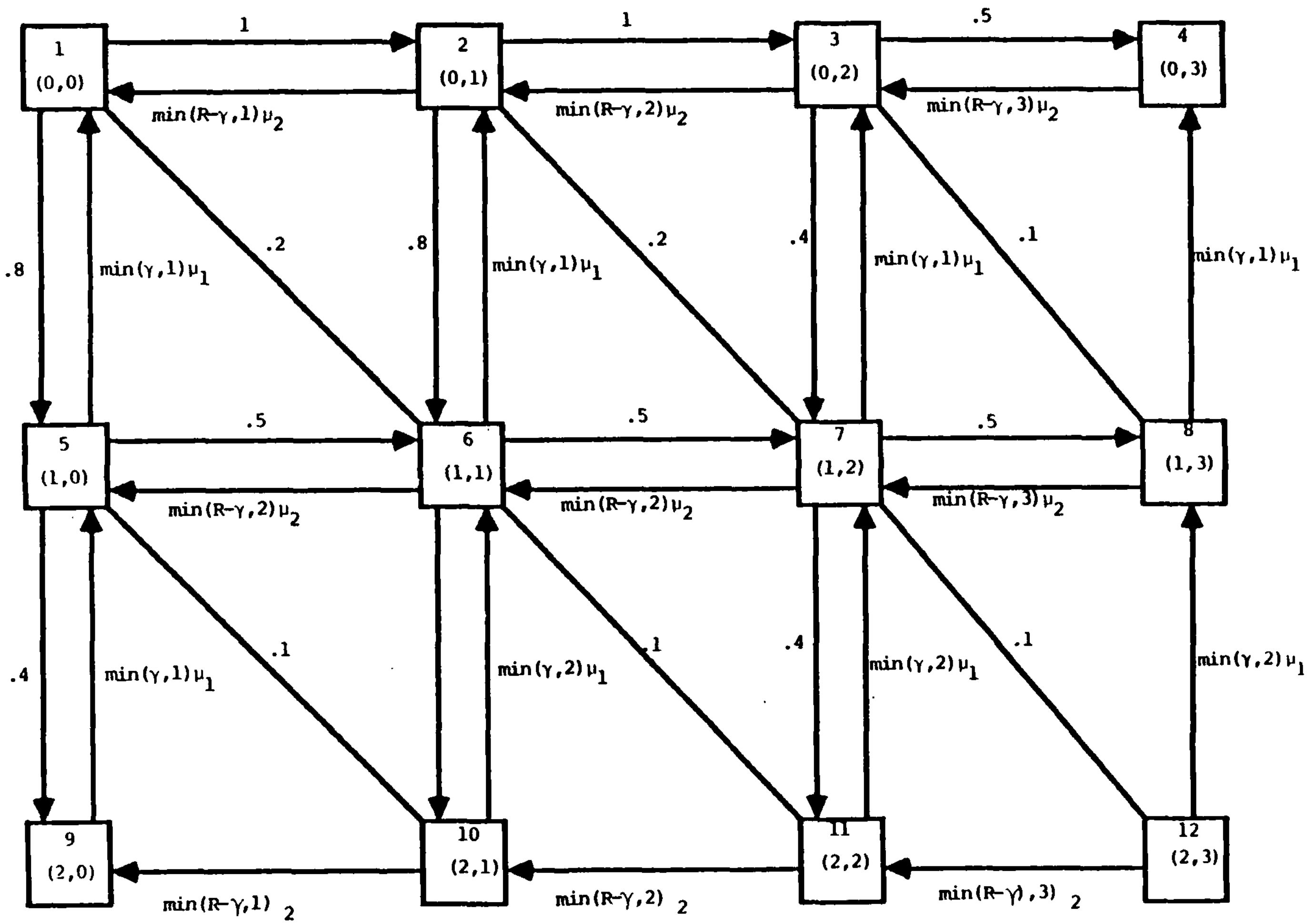

Figure 3.5. An Illustration of Transition Flow as a Function of Decision Taken at Each State

$$
Q_o = \begin{array}{rl}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{rl} 1 & (0,0) \\ 2 & (0,1) \\ 3 & (0,2) \\ 4 & (0,3) \\ 5 & (1,0) \\ 6 & (1,1) \\ 7 & (1,2) \\ 8 & (1,3) \\ 9 & (2,0) \\ 10 & (2,1) \\ 11 & (2,2) \\ 12 & (2,3) \end{array} &
\begin{bmatrix}
-2 & 1 & & & .8 & .2 & & & & & & \\
1 & -3 & 1 & & & .8 & .2 & & & & & \\
 & 2 & -3 & .5 & & & .4 & .1 & & & & \\
 & & 3 & -3 & & & & & & & & \\
 & & & & -1 & .5 & & & .4 & .1 & & \\
 & & & & 1 & -2 & .5 & & & .4 & .1 & \\
 & & & & & 2 & -3 & .5 & & & .4 & .1 \\
 & & & & & & 3 & -3 & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & 1 & -1 & & \\
 & & & & & & & & & 2 & -2 & \\
 & & & & & & & & & & 3 & -3
\end{bmatrix}
\end{array}
$$

Figure 3.6. The Infinitesimal Generator for the Repair Process with $R_1 = 0$ and $R_2 = 3$

$$
Q_1 = \begin{array}{ccc}
 & & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{array}{c} (0,0) \\ (0,1) \\ (0,2) \\ (0,3) \\ (1,0) \\ (1,1) \\ (1,2) \\ (1,3) \\ (2,0) \\ (3,1) \\ (3,2) \\ (2,3) \end{array} & \left[\begin{array}{cccccccccccc}
-2 & 1 & & & .8 & .2 & & & & & & \\
1 & -3 & 1 & & & .8 & .2 & & & & & \\
 & 2 & -3 & .5 & & & .4 & .1 & & & & \\
 & & 2 & -2 & & & & & & & & \\
.5 & & & & -1.5 & .5 & & & .4 & .1 & & \\
 & .5 & & & 1 & -2.5 & .5 & & & .4 & .1 & \\
 & & .5 & & & 2 & -3.5 & .5 & & & .4 & .1 \\
 & & & .5 & & & 2 & -2.5 & & & & \\
 & & & & .5 & & & & -.5 & & & \\
 & & & & & .5 & & & 1 & -1.5 & & \\
 & & & & & & .5 & & & 2 & -2.5 & \\
 & & & & & & & .5 & & & 2 & -2.5
\end{array}\right]
\end{array}
$$

Figure 3.7. The Infinitesimal Generator for the Repair Process with $R_1 = 1$ and $R_2 = 2$

$$
Q_2 = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{rc} 1 & (0,0) \\ 2 & (0,1) \\ 3 & (0,2) \\ 4 & (0,3) \\ 5 & (1,0) \\ 6 & (1,1) \\ 7 & (1,2) \\ 8 & (1,3) \\ 9 & (2,0) \\ 10 & (2,1) \\ 11 & (2,2) \\ 12 & (2,3) \end{array} &
\begin{bmatrix}
-2 & 1 & & & .8 & .2 & & & & & & \\
1 & -3 & 1 & & & .8 & .2 & & & & & \\
 & 1 & -2 & .5 & & & .4 & .1 & & & & \\
 & & 1 & -1 & & & & & & & & \\
.5 & & & & -1.5 & .5 & & & .4 & .1 & & \\
 & .5 & & & 1 & -2.5 & .5 & & & .4 & .1 & \\
 & & .5 & & & 1 & -2.5 & .5 & & & .4 & .1 \\
 & & & .5 & & & 1 & -1.5 & & & & \\
 & & & & 1 & & & & -1 & & & \\
 & & & & & 1 & & & 1 & -2 & & \\
 & & & & & & 1 & & & 1 & -2 & \\
 & & & & & & & 1 & & & 1 & -2
\end{bmatrix}
\end{array}
$$

Figure 3.8. The Infinitesimal Generator for the Repair Process with $R_1 = 2$ and $R_2 = 1$

$$
Q_3 = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{c} 1\ (0,0) \\ 2\ (0,1) \\ 3\ (0,2) \\ 4\ (0,3) \\ 5\ (1,0) \\ 6\ (1,1) \\ 7\ (1,2) \\ 8\ (1,3) \\ 9\ (2,0) \\ 10\ (2,1) \\ 11\ (2,2) \\ 12\ (2,3) \end{array} &
\begin{bmatrix}
-2 & 1 & & & .8 & .2 & & & & & & \\
 & -2 & 1 & & & .8 & .2 & & & & & \\
 & & -1 & .5 & & & .4 & .1 & & & & \\
 & & & & & & & & & & & \\
.5 & & & & -1.5 & .5 & & & .4 & .1 & & \\
 & .5 & & & & -1.5 & .5 & & & .4 & .1 & \\
 & & .5 & & & & -1.5 & .5 & & & .4 & .1 \\
 & & & .5 & & & & -.5 & & & & \\
 & & & & 1 & & & & -1 & & & \\
 & & & & & 1 & & & & -1 & & \\
 & & & & & & 1 & & & & -1 & \\
 & & & & & & & 1 & & & & -1
\end{bmatrix}
\end{array}
$$

Figure 3.9. The Infinitesimal Generator for the Repair Process with $R_1 = 3$ and $R_2 = 0$

needs to be determined. The repair policy represented by the vector γ recommends that: whenever the system is in state n take the decision $\gamma(n)$, (e.g., if in state 1 then take decision 1, if in state 5 take decision 3, etc.). Now, to determine the corresponding infinitesimal generator Q, we use the following algorithm:

For: $n = 1, (M_1+1)(M_2+1)$

$$Q(n,m) = Q_{\gamma(n)}(n,m) \qquad m = 1,\ldots,(M_1+1)(M_2+1).$$

The preceding discussion for determining Q will be clearer when the infinitesimal generator Q^* for the optimal policy is determined.

Note that the policy decision is made potentially at <u>each</u> state change.

In an attempt to search for the most favorable (i.e., optimal) policy that needs to be followed in order to maximize the readiness of the squadron we maximize the expected number of aircraft ready to engage in combat. Table 3.2 shows the total number of aircraft ready to engage (f(n)) for the various states of the process.

TABLE 3.2

NUMBER OF AIRCRAFT OPERATIONAL WHEN THE PROCESS IS IN STATE (I,J)

state: n	state: (i,j)	No. of Aircraft f(n)
1	(0,0)	2
2	(0,1)	2
3	(0,2)	1
4	(0,3)	0
5	(1,0)	1
6	(1,1)	1
7	(1,2)	1
8	(1,3)	0
9	(2,0)	0
10	(2,1)	0
11	(2,2)	0
12	(2,3)	0

K. RESULTS AND ANALYSIS

1. Steady State Analysis

For any policy the long-run expected number of aircraft that are up and ready to engage per unit time, $E[D(\infty)]$, can be calculated by multiplying the number of aircraft up when in state n (f(n)) by the long-run (limiting) probability of being in state n (π_n) and summing over all possible states.

$$E[D(\infty)] = \sum_{n=1}^{(M_1+1)(M_2+1)} f(n)\pi_n(\gamma) \qquad \forall \gamma$$

$$E[D(\infty)] = \sum_{n=1}^{12} f(n)\pi_n(\gamma) \qquad \forall \gamma$$

where:

$\{\pi_1(\gamma),...,\pi_{12}(\gamma)\}$ represents the steady-state distribution of the state of the system under the policy being evaluated.

The state space consists of only 12 states.

Since all states communicate, the process is a regular Markov process. Therefore the limiting probabilities exist. The resulting optimal policy is dependent on the number of aircraft, modules of Type 1, modules of Type 2, and repairmen assigned to the base.

2. Optimization Results

a. Linear Programming Results

The results of applying the linear programming optimization approach using the LINDO package are shown below in Table 3.3. The resulting optimal policy is shown immediately following Table 3.3.

This optimal policy recommends assigning no repairmen to Shop 1 when there are no items of Type 1 down, assigning one repairman to Shop 1 when there is an item of Type 1 down, and assigning two repairmen to Shop 1 when there

TABLE 3.3

LINEAR PROGRAMMING RESULTS

$\pi_0(1)$	=	.1180
$\pi_0(2)$	=	.0990
$\pi_0(3)$	=	.0511
$\pi_0(4)$	=	.0105
$\pi_3(5)$	=	.2739
$\pi_2(6)$	=	.1536
$\pi_1(7)$	=	.0454
$\pi_1(8)$	=	.0120
$\pi_3(9)$	=	.1629
$\pi_2(10)$	=	.0534
$\pi_2(11)$	=	.0179
$\pi_2(12)$	=	.0023

	1	2	3	4	5	6	7	8	9	10	11	12
State	(0,0)	(0,1)	(0,2)	(0,3)	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)
Decision	0	0	0	0	3	2	1	1	3	2	2	2

are two items of Type 1 down. The remaining number of repairmen are to be assigned to Shop 2 if there are items of Type 2 down or they are idle. Following this policy gives an objective function value (i.e., expected number of aircraft that are operational) of .9579. This policy determines a unique bivariate-continuous-time Markov process with the following infinitesimal generator:

$$
Q^* = \begin{array}{cc}
 & \begin{array}{cccccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{c} (0,0) \\ (0,1) \\ (0,2) \\ (0,3) \\ (1,0) \\ (1,1) \\ (1,2) \\ (1,3) \\ (2,0) \\ (2,1) \\ (2,2) \\ (2,3) \end{array} &
\begin{bmatrix}
-2 & 1 & & & .8 & .2 & & & & & & \\
1 & -3 & 1 & & & .8 & .2 & & & & & \\
 & 2 & -3 & .5 & & & .4 & .1 & & & & \\
 & & 3 & -3 & & & & & & & & \\
.5 & & & & -1.5 & .5 & & & .4 & .1 & & \\
 & .5 & & & 1 & -2.5 & .5 & & & .4 & .1 & \\
 & & .5 & & & 2 & -3.5 & .5 & & & .4 & .1 \\
 & & & .5 & & & 2 & -2.5 & & & & \\
 & & & & 1 & & & & -1 & & & \\
 & & & & & 1 & & & 1 & -2 & & \\
 & & & & & & 1 & & & 1 & -2 & \\
 & & & & & & & 1 & & & 1 & -2
\end{bmatrix}
\end{array}
$$

Therefore, by assigning the repairmen according to the optimal policy, the resulting stochastic process can be assumed to be a bivariate-continuous-time-Markov proces $\{X_1(t), X_2(t); t \geq 0\}$, over the state space $S = \{(0,0),(0,1),(0,2),\ldots,(2,3)\}$, with the infinitesimal generator Q* and initial conditions:

$$P_{n,m}(o) = \begin{cases} 1 & \text{if } n = 0,\ m = 0 \\ 0 & \text{otherwise} . \end{cases}$$

Such a process has the limiting probabilities shown in Table 3.3 and an expected number of aircraft ready to engage of .9579.

b. Dynamic Programming Results

The results of applying the dynamic programming procedure are given in Table 3.4. The optimal policy is shown immediately following Table 3.4. The total objective function value = .9579.

The infinitesimal generator (Q*) for the continuous-time Markov process resulting from the application of the optimal policy is:

TABLE 3.4

DYNAMIC PROGRAMMING RESULTS

$\pi_3(1)$	=	.1180
$\pi_2(2)$	=	.0990
$\pi_1(3)$	=	.0511
$\pi_0(4)$	=	.0105
$\pi_3(5)$	=	.2739
$\pi_2(6)$	=	.1536
$\pi_1(7)$	=	.0454
$\pi_1(8)$	=	.0120
$\pi_3(9)$	=	.1629
$\pi_2(10)$	=	.0534
$\pi_2(11)$	=	.0179
$\pi_2(12)$	=	.0023

The optimal policy is:

	1	2	3	4	5	6	7	8	9	10	11	12
State:	(0,0)	(0,1)	(0,2)	(0,3)	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)
Decision:	3	2	1	0	3	2	1	1	3	2	2	2

$$
Q^* = \begin{array}{cc}
 & \begin{array}{cccccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) \end{array} \\
\begin{array}{c} (0,0) \\ (0,1) \\ (0,2) \\ (0,3) \\ (1,0) \\ (1,1) \\ (1,2) \\ (1,3) \\ (2,0) \\ (2,1) \\ (2,2) \\ (2,3) \end{array} &
\left[\begin{array}{cccccccccccc}
-2 & 1 & & & .8 & .2 & & & & & & \\
1 & -3 & 1 & & & .8 & .2 & & & & & \\
 & 2 & -3 & .5 & & & .4 & .1 & & & & \\
 & & 3 & -3 & & & & & & & & \\
.5 & & & & -1.5 & .5 & & & .4 & .1 & & \\
 & .5 & & & 1 & -2.5 & .5 & & & .4 & .1 & \\
 & & .5 & & & 2 & -3.5 & .5 & & & .4 & .1 \\
 & & & .5 & & & 2 & -2.5 & & & & \\
 & & & & 1 & & & & -1 & & & \\
 & & & & & 1 & & & 1 & -2 & & \\
 & & & & & & 1 & & & 1 & -2 & \\
 & & & & & & & 1 & & & 1 & -2
\end{array}\right]
\end{array}
$$

Suppose the squadron is assigned 6 aircraft, $M_1 = 6$ modules of Type 1, and $M_2 = 7$ modules of Type 2. The repair shop supervisor is assigned 6 repairmen, while the remaining parameters are the same as in the previous example. Table (3.5) gives a summary of the results. It shows for every state 'n', the corresponding (i,j) state for the bivariate process, the recommended number of repairmen to be assigned to Shop 1, and the number of aircraft that are up and ready to engage whenever the process is found to be in state (i,j). Following this policy the squadron will maintain on the average 2.88 aircraft up at any time.

L. COMPARISON OF THE DYNAMIC PROGRAMMING AND THE LINEAR PROGRAMMING SOLUTIONS

A comparison of the dynamic programming and the linear programming approaches shows that the two approaches yield different policies. Table 3.6 shows the results side by side.

Both approaches gave the same limiting distribution and hence the same objective value. The difference in the optimal policies when $\{X_1(\infty) = 0\}$ suggests that there is more than one optimal policy, i.e., the optimal policy is not unique. This is intuitively obvious since there is no cost associated with moving repairmen between shops, the assignment of idle repairmen to Shop 1 versus Shop 2 doesn't affect the objective function value.

Both approaches resulted in the same infinitesimal generator Q*, and since both processes have the same initial

TABLE 3.5

A SUMMARY OF THE RESULTS OBTAINED BY DYNAMIC PROGRAMMING RUN FOR THE 6 AIRCRAFT CASE

N	(I.J)	DECISION	AIRCRAFT UP
1	(0.0)	6.	6.
2	(0.1)	5.	6.
3	(0.2)	4.	5.
4	(0.3)	3.	4.
5	(0.4)	2.	3.
6	(0.5)	1.	2.
7	(0.6)	0.	1.
8	(0.7)	0.	0.
9	(1.0)	6.	5.
10	(1.1)	5.	5.
11	(1.2)	4.	5.
12	(1.3)	3.	4.
13	(1.4)	2.	3.
14	(1.5)	1.	2.
15	(1.6)	0.	1.
16	(1.7)	0.	0.
17	(2.0)	6.	4.
18	(2.1)	5.	4.
19	(2.2)	4.	4.
20	(2.3)	3.	4.
21	(2.4)	2.	3.
22	(2.5)	1.	2.
23	(2.6)	0.	1.
24	(2.7)	0.	0.
25	(3.0)	6.	3.
26	(3.1)	5.	3.
27	(3.2)	4.	3.
28	(3.3)	3.	3.
29	(3.4)	3.	3.
30	(3.5)	3.	2.
31	(3.6)	0.	1.
32	(3.7)	0.	0.
33	(4.0)	6.	2.
34	(4.1)	5.	2.
35	(4.2)	4.	2.
36	(4.3)	4.	2.
37	(4.4)	4.	2.
38	(4.5)	4.	2.
39	(4.6)	0.	1.
40	(4.7)	0.	0.
41	(5.0)	6.	1.
42	(5.1)	5.	1.
43	(5.2)	5.	1.
44	(5.3)	5.	1.
45	(5.4)	5.	1.
46	(5.5)	5.	1.
47	(5.6)	5.	1.
48	(5.7)	0.	0.
49	(6.0)	6.	0.
50	(6.1)	6.	0.
51	(6.2)	6.	0.
52	(6.3)	6.	0.
53	(6.4)	6.	0.
54	(6.5)	6.	0.
55	(6.6)	6.	0.
56	(6.7)	6.	0.

TABLE 3.6

RESULTS OF BOTH OPTIMIZATION APPROACHES SIDE BY SIDE

State	Decision: D-P	Decision: L-P
(0,0)	3	0
(0,1)	2	0
(0,2)	1	0
(0,3)	0	0
(1,0)	3	3
(1,1)	2	2
(1,2)	1	1
(1,3)	1	1
(2,0)	3	3
(2,1)	2	2
(2,2)	2	2
(2,3)	2	2

distribution, therefore, it may be concluded that both approaches identified the same bivariate-continuous-time Markov process.

Observe that whenever the process is in a state (i,j) such that $i+j \leq R$, the optimal decision will always be to assign i repairmen to Shop 1 and j repairmen to Shop 2 and any idle repairmen to either shop. However, if $i+j > R$, then a decision has to be made to determine the number of repairmen to be assigned to Shop 1.

Consider the state (2,2). The optimal policy calls for assigning 2 repairmen to Shop 1, and a single repairman to Shop 2. This shows that the supervisor is giving a priority to items of Type 1 since both aircraft are down, and only one item of Type 1 is needed to get one aircraft up.

Consider the state (2,3). The optimal policy still calls for the assignment of 2 repairmen to Shop 1, and a single repairman to Shop 2. This shows that the supervisor must still give priority to items of Type 1, even though an aircraft requires both items to be in up condition. This is due to the difference in the repair rates, $\mu_1 = .5$ and $\mu_2 = 1$ module per day. Therefore, it takes two days on the average to repair a module of Type 1 and one day to repair a module of Type 2. To carry out the analysis mathematically, define:

U_{1i} = Time to repair the i^{th} item of Type 1, $i = 1,2$.

U_1 = Time required to repair the first item of Type 1.

$$U_1 = \min\{U_{11}, U_k\}$$

Therefore, U_{1i}, $i = 1,2$ are two independent identically distributed exponential random variables with rate μ_1.

Hence:

$$
\begin{aligned}
P\{U_1 > t\} &= P\{\min(U_{11},U_{12}) > t\} \\
&= P\{U_{11} > t, U_{12} > t\} \\
&= P\{U_{11} > t\}P\{U_{12} > t\} \\
&= e^{-\mu_1 t} \cdot e^{-\mu_1 t} = e^{-2\mu_1 t}
\end{aligned}
$$

Thus: $U_1 = \min\{U_{11},U_{12}\}$ is an exponential random variable with rate $2\mu_1$. In the example, $\mu_1 = .5$. Therefore the rate to get the first item of Type 1, when the process is in state (2,3) with 2 repairmen assigned to Shop 1, is 1. Therefore, on the average, when in state (2,3) (i.e., all modules are down and 2 modules are needed, one of each type, to get an aircraft up), it takes on the average one day to get the first aircraft operational. If the supervisor has assigned only one repairman to Shop 1 and 2 to Shop 2, then it will take on the average 2 days to get a module of Type 1 up. By following the above analysis for Type 2 modules, it will take on the average 1/2 day to get the first module. However, an aircraft requires both types to be considered up. Therefore on the average it will take 2 days to get the first aircraft up. The preceding analysis supports the decision provided by the optimal policy.

As shown earlier, the decisions to be taken whenever the process is in a state (i,j) such that $i+j \leq R$ ($R = 3$) is trivial. To determine if the resulting repair

policy is indeed optimal, a Fortran program was written to evaluate all possible policies. This is done by considering all possible decisions that could be taken whenever the process is found in state: (1,3), (2,2), or (2,3). Table (3.7) shows all possible policies, the corresponding limiting probabilities of the resulting process and the objective function value. It is clear that the repair policy suggested by both the linear programming and the dynamic programming approaches are optimal policies, resulting in the same Markov process.

The linear programming approach has the advantage of the widespread availability of commercial packages for use (e.g., LINDO and MPS III). An additional advantage of the linear programming approach is the ability to easily conduct sensitivity analyses. However, for large numbers of modules, aircraft and repairmen, the linear programming approach may be impractical. It seems that the dynamic programming approach is more suitable for the larger size problems because:

- It requires less data input and preparation.
- There might be a possibility of degenerate solutions in linear programming which will yield an impractical optimal solution (e.g., in the case when there are round-off errors in calculating the rates such that

$$\sum_{j=1}^{(M_1+1)(M_2+1)} q_\gamma(i,j)$$

  doesn't equal 1 exactly).

TABLE 3.7

ALL POSSIBLE NONTRIVIAL POLICIES

POLICY	LIMITING PROBABILITIES												EXPECTED A/C
3 2 1 0 3 2 1 0 3 2 0 0	0.1175	0.0951	0.0459	0.0077	0.2799	0.1520	0.0395	0.0081	0.1730	0.0610	0.0174	0.0013	0.9426
3 2 1 0 3 2 1 0 3 2 0 1	0.1175	0.0952	0.0460	0.0077	0.2797	0.1521	0.0396	0.0084	0.1734	0.0615	0.0171	0.0016	0.9430
3 2 1 0 3 2 1 0 3 2 0 2	0.1175	0.0953	0.0461	0.0077	0.2794	0.1523	0.0401	0.0089	0.1729	0.0611	0.0166	0.0020	0.9437
3 2 1 0 3 2 1 0 3 2 0 3	0.1173	0.0954	0.0463	0.0077	0.2786	0.1524	0.0400	0.0097	0.1716	0.0602	0.0158	0.0041	0.9436
3 2 1 0 3 2 1 0 3 2 1 0	0.1180	0.0963	0.0470	0.0078	0.2796	0.1528	0.0423	0.0086	0.1712	0.0593	0.0146	0.0014	0.9513
3 2 1 0 3 2 1 0 3 2 1 1	0.1180	0.0964	0.0471	0.0078	0.2794	0.1530	0.0426	0.0089	0.1708	0.0591	0.0143	0.0017	0.9517
3 2 1 0 3 2 1 0 3 2 1 2	0.1180	0.0964	0.0472	0.0079	0.2791	0.1540	0.0430	0.0095	0.1703	0.0586	0.0139	0.0021	0.9521
3 2 1 0 3 2 1 0 3 2 1 3	0.1178	0.0965	0.0473	0.0079	0.2782	0.1540	0.0436	0.0103	0.1691	0.0578	0.0131	0.0044	0.9516
3 2 1 0 3 2 1 0 3 2 2 0	0.1177	0.0973	0.0483	0.0081	0.2763	0.1552	0.0469	0.0094	0.1651	0.0546	0.0195	0.0016	0.9568
3 2 1 0 3 2 1 0 3 2 2 1	0.1177	0.0974	0.0484	0.0081	0.2762	0.1552	0.0471	0.0098	0.1649	0.0544	0.0191	0.0019	0.9570
3 2 1 0 3 2 1 0 3 2 2 2	0.1177	0.0974	0.0484	0.0081	0.2760	0.1553	0.0474	0.0103	0.1645	0.0541	0.0186	0.0024	0.9574
3 2 1 0 3 2 1 0 3 2 2 3	0.1175	0.0974	0.0485	0.0081	0.2752	0.1552	0.0478	0.0112	0.1635	0.0535	0.0173	0.0048	0.9565
3 2 1 0 3 2 1 0 3 2 3 0	0.1154	0.0979	0.0502	0.0084	0.2659	0.1557	0.0552	0.0109	0.1508	0.0444	0.0432	0.0018	0.9538
3 2 1 0 3 2 1 0 3 2 3 1	0.1155	0.0979	0.0502	0.0084	0.2660	0.1558	0.0553	0.0113	0.1509	0.0445	0.0421	0.0022	0.9542
3 2 1 0 3 2 1 0 3 2 3 2	0.1155	0.0980	0.0503	0.0084	0.2662	0.1559	0.0553	0.0118	0.1509	0.0445	0.0405	0.0028	0.9547
3 2 1 0 3 2 1 0 3 2 3 3	0.1154	0.0979	0.0502	0.0084	0.2659	0.1557	0.0552	0.0127	0.1508	0.0444	0.0377	0.0055	0.9538
3 2 1 0 3 2 1 1 3 2 0 0	0.1178	0.0964	0.0481	0.0096	0.2782	0.1507	0.0380	0.0095	0.1724	0.0611	0.0170	0.0013	0.9433
3 2 1 0 3 2 1 1 3 2 0 1	0.1178	0.0965	0.0482	0.0097	0.2780	0.1508	0.0382	0.0099	0.1719	0.0608	0.0167	0.0015	0.9438
3 2 1 0 3 2 1 1 3 2 0 2	0.1178	0.0967	0.0485	0.0098	0.2776	0.1509	0.0386	0.0104	0.1715	0.0603	0.0162	0.0019	0.9445
3 2 1 0 3 2 1 1 3 2 0 3	0.1176	0.0969	0.0489	0.0100	0.2766	0.1508	0.0392	0.0114	0.1700	0.0594	0.0154	0.0039	0.9445
3 2 1 0 3 2 1 1 3 2 1 0	0.1183	0.0976	0.0492	0.0099	0.2778	0.1523	0.0408	0.0101	0.1697	0.0584	0.0143	0.0014	0.9519
3 2 1 0 3 2 1 1 3 2 1 1	0.1183	0.0977	0.0494	0.0100	0.2775	0.1524	0.0410	0.0105	0.1693	0.0583	0.0140	0.0016	0.9523
3 2 1 0 3 2 1 1 3 2 1 2	0.1182	0.0979	0.0496	0.0101	0.2771	0.1524	0.0413	0.0111	0.1687	0.0579	0.0135	0.0021	0.9528
3 2 1 0 3 2 1 1 3 2 1 3	0.1180	0.0980	0.0500	0.0103	0.2761	0.1523	0.0418	0.0120	0.1675	0.0570	0.0128	0.0042	0.9523
3 2 1 0 3 2 1 1 3 2 2 0	0.1180	0.0987	0.0507	0.0103	0.2744	0.1536	0.0451	0.0111	0.1637	0.0539	0.0190	0.0015	0.9573
3 2 1 0 3 2 1 1 3 2 2 1	0.1180	0.0988	0.0509	0.0104	0.2742	0.1536	0.0452	0.0114	0.1634	0.0537	0.0185	0.0018	0.9576
3 2 1 0 3 2 1 1 3 2 2 2	0.1180	0.0990	0.0511	0.0105	0.2739	0.1536	0.0454	0.0120	0.1629	0.0534	0.0179	0.0023	0.9579
3 2 1 0 3 2 1 1 3 2 2 3	0.1178	0.0991	0.0514	0.0107	0.2730	0.1534	0.0457	0.0130	0.1619	0.0527	0.0168	0.0046	0.9571
3 2 1 0 3 2 1 1 3 2 3 0	0.1158	0.0995	0.0529	0.0109	0.2641	0.1539	0.0529	0.0127	0.1496	0.0440	0.0419	0.0018	0.9545
3 2 1 0 3 2 1 1 3 2 3 1	0.1158	0.0996	0.0531	0.0110	0.2641	0.1539	0.0529	0.0131	0.1496	0.0440	0.0408	0.0021	0.9548
3 2 1 0 3 2 1 1 3 2 3 2	0.1159	0.0998	0.0532	0.0112	0.2642	0.1539	0.0528	0.0137	0.1496	0.0440	0.0391	0.0026	0.9554
3 2 1 0 3 2 1 1 3 2 3 3	0.1158	0.0998	0.0534	0.0114	0.2638	0.1536	0.0526	0.0148	0.1494	0.0439	0.0364	0.0053	0.9546
3 2 1 0 3 2 1 2 3 2 0 0	0.1173	0.0967	0.0492	0.0108	0.2757	0.1491	0.0369	0.0156	0.1705	0.0603	0.0167	0.0012	0.9389
3 2 1 0 3 2 1 2 3 2 0 1	0.1173	0.0969	0.0494	0.0109	0.2753	0.1491	0.0371	0.0162	0.1701	0.0599	0.0164	0.0015	0.9392
3 2 1 0 3 2 1 2 3 2 0 2	0.1172	0.0971	0.0497	0.0111	0.2748	0.1491	0.0374	0.0170	0.1694	0.0594	0.0159	0.0019	0.9396
3 2 1 0 3 2 1 2 3 2 0 3	0.1171	0.0973	0.0502	0.0114	0.2736	0.1489	0.0378	0.0185	0.1679	0.0585	0.0150	0.0038	0.9392
3 2 1 0 3 2 1 2 3 2 1 0	0.1177	0.0980	0.0504	0.0112	0.2751	0.1506	0.0396	0.0166	0.1678	0.0578	0.0139	0.0013	0.9471
3 2 1 0 3 2 1 2 3 2 1 1	0.1177	0.0981	0.0506	0.0113	0.2747	0.1505	0.0397	0.0172	0.1674	0.0575	0.0137	0.0016	0.9472
3 2 1 0 3 2 1 2 3 2 1 2	0.1177	0.0982	0.0509	0.0115	0.2742	0.1505	0.0400	0.0181	0.1667	0.0570	0.0132	0.0020	0.9476
3 2 1 0 3 2 1 2 3 2 1 3	0.1174	0.0984	0.0513	0.0118	0.2730	0.1502	0.0403	0.0195	0.1653	0.0561	0.0125	0.0040	0.9465
3 2 1 0 3 2 1 2 3 2 2 0	0.1174	0.0991	0.0520	0.0117	0.2716	0.1516	0.0437	0.0180	0.1618	0.0532	0.0185	0.0015	0.9519
3 2 1 0 3 2 1 2 3 2 2 1	0.1176	0.0992	0.0522	0.0118	0.2713	0.1516	0.0438	0.0187	0.1614	0.0529	0.0181	0.0018	0.9519
3 2 1 0 3 2 1 2 3 2 2 2	0.1176	0.0993	0.0524	0.0120	0.2708	0.1515	0.0439	0.0196	0.1609	0.0526	0.0176	0.0022	0.9520
3 2 1 0 3 2 1 2 3 2 2 3	0.1171	0.0994	0.0528	0.0123	0.2697	0.1511	0.0440	0.0211	0.1598	0.0519	0.0166	0.0044	0.9507
3 2 1 0 3 2 1 2 3 2 3 0	0.1152	0.0999	0.0543	0.0125	0.2611	0.1516	0.0511	0.0207	0.1478	0.0434	0.0407	0.0017	0.9483
3 2 1 0 3 2 1 2 3 2 3 1	0.1152	0.1000	0.0545	0.0126	0.2610	0.1516	0.0510	0.0213	0.1478	0.0434	0.0396	0.0020	0.9485
3 2 1 0 3 2 1 2 3 2 3 2	0.1153	0.1001	0.0547	0.0128	0.2609	0.1514	0.0508	0.0223	0.1477	0.0433	0.0380	0.0025	0.9487
3 2 1 0 3 2 1 2 3 2 3 3	0.1152	0.1002	0.0549	0.0131	0.2603	0.1510	0.0505	0.0239	0.1473	0.0432	0.0353	0.0050	0.9475
3 2 1 0 3 2 1 3 3 2 0 0	0.1151	0.0902	0.0545	0.0162	0.2662	0.1410	0.0321	0.0429	0.1623	0.0567	0.0151	0.0011	0.9190
3 2 1 0 3 2 1 3 3 2 0 1	0.1151	0.0904	0.0546	0.0165	0.2635	0.1416	0.0321	0.0443	0.1617	0.0563	0.0148	0.0013	0.9186
3 2 1 0 3 2 1 3 3 2 0 2	0.1149	0.0904	0.0551	0.0169	0.2626	0.1412	0.0321	0.0443	0.1607	0.0557	0.0143	0.0016	0.9181
3 2 1 0 3 2 1 3 3 2 0 3	0.1146	0.0909	0.0559	0.0176	0.2606	0.1405	0.0321	0.0497	0.1588	0.0546	0.0136	0.0032	0.9161
3 2 1 0 3 2 1 3 3 2 1 0	0.1154	0.0994	0.0557	0.0168	0.2658	0.1427	0.0342	0.0454	0.1594	0.0542	0.0126	0.0011	0.9252
3 2 1 0 3 2 1 3 3 2 1 1	0.1152	0.0995	0.0560	0.0171	0.2623	0.1426	0.0342	0.0460	0.1588	0.0539	0.0123	0.0014	0.9247
3 2 1 0 3 2 1 3 3 2 1 2	0.1152	0.0997	0.0565	0.0176	0.2616	0.1420	0.0341	0.0480	0.1570	0.0533	0.0118	0.0017	0.9239
3 2 1 0 3 2 1 3 3 2 1 3	0.1148	0.1000	0.0572	0.0182	0.2596	0.1412	0.0340	0.0523	0.1541	0.0523	0.0111	0.0034	0.9215
3 2 1 0 3 2 1 3 3 2 2 0	0.1150	0.1005	0.0575	0.0177	0.2589	0.1430	0.0375	0.0490	0.1536	0.0490	0.0165	0.0012	0.9278
3 2 1 0 3 2 1 3 3 2 2 1	0.1149	0.1006	0.0578	0.0180	0.2583	0.1427	0.0373	0.0504	0.1528	0.0495	0.0161	0.0015	0.9271
3 2 1 0 3 2 1 3 3 2 2 2	0.1147	0.1008	0.0583	0.0185	0.2574	0.1423	0.0372	0.0525	0.1520	0.0491	0.0155	0.0019	0.9261
3 2 1 0 3 2 1 3 3 2 2 3	0.1144	0.1010	0.0590	0.0192	0.2555	0.1413	0.0368	0.0560	0.1505	0.0483	0.0144	0.0037	0.9236
3 2 1 0 3 2 1 3 3 2 3 0	0.1127	0.1014	0.0602	0.0192	0.2481	0.1419	0.0432	0.0552	0.1400	0.0408	0.0350	0.0014	0.9216
3 2 1 0 3 2 1 3 3 2 3 1	0.1127	0.1015	0.0605	0.0195	0.2477	0.1416	0.0429	0.0567	0.1398	0.0407	0.0347	0.0017	0.9218
3 2 1 0 3 2 1 3 3 2 3 2	0.1126	0.1017	0.0609	0.0200	0.2471	0.1411	0.0424	0.0588	0.1394	0.0406	0.0332	0.0021	0.9202
3 2 1 0 3 2 1 3 3 2 3 3	0.1124	0.1018	0.0615	0.0204	0.2458	0.1402	0.0416	0.0625	0.1386	0.0403	0.0307	0.0042	0.9175

IV. FORMULATION AND ANALYSIS OF SOME COMBAT MODELS

A. INTRODUCTION

In order to formulate Combat-Logistics models, consideration of some relevant combat situations are necessary. This chapter presents an analysis of an air defense scenario for a relatively small country, or, alternatively, an area (or sector) of responsibility to a defense system. The air defense system is assumed to consist of:

1. An interceptor squadron made up of a relatively small number of aircraft equipped to engage in air-to-air combat.
2. A Surface-to-Air system that consists of a number of Surface-to-Air Missile (SAM) batteries, and Anti-Aircraft-Artillary (AAA) batteries.
3. An early warning system that has a long range radar, and uses information from an airborne warning and control system (AWACS) operated by friendly forces.
4. A command, control and communication (C^3) system.

The air defense system is responsible for defending against any hostile air attack from a hostile agent, e.g., nation. Suppose that the objective of the air defense system is to absorb the initial strike; after the initial campaign it is assumed that there will be some support from other friendly agents. The term "absorb" means to cause maximum possible reduction of the hostile damage to the area being defended. Consider the following threat scenario that the air defense system might encounter. A hostile attack occurs

unexpectedly with multiple bombers; the attack objective is to destroy some high-valued military and/or national targets. The bombers appear in a wave, i.e., essentially simultaneously.

It is assumed that the air defense doctrine dictates that for any incoming threat, the air interceptor squadron will first engage the threat until the threat reaches a certain imaginary line (set by doctrine); surviving bombers that cross the line are attacked by the SAM units. The imaginary line will be referred to as the hand-over line, and the term "threat" means the enemy bombers. Therefore, the time required by the threat to reach the hand-over line from time of detection determines the time that is possible for the air interceptors to engage the threat. At a later time (or shorter range) the SAM units will engage until the threat reaches yet another hand-over line, at which point the AAA system will take over to engage the surviving threat elements. At each line the leakage from the stage before must be dealt with by the system in question. Figure (4.1) illustrates the above scenario.

Notice that each stage constitutes a different type of combat; the leakage from that stage provides the input for the stage following. For example, the leakage of the air-to-air combat process is the input to the SAM-to-bombers engagement process (e.g., the number of bombers to be engaged by SAM).

A feasible enemy doctrine dictates that the mission of the bombers is to be aborted as soon as the number of bombers surviving the combat drops below a certain number.

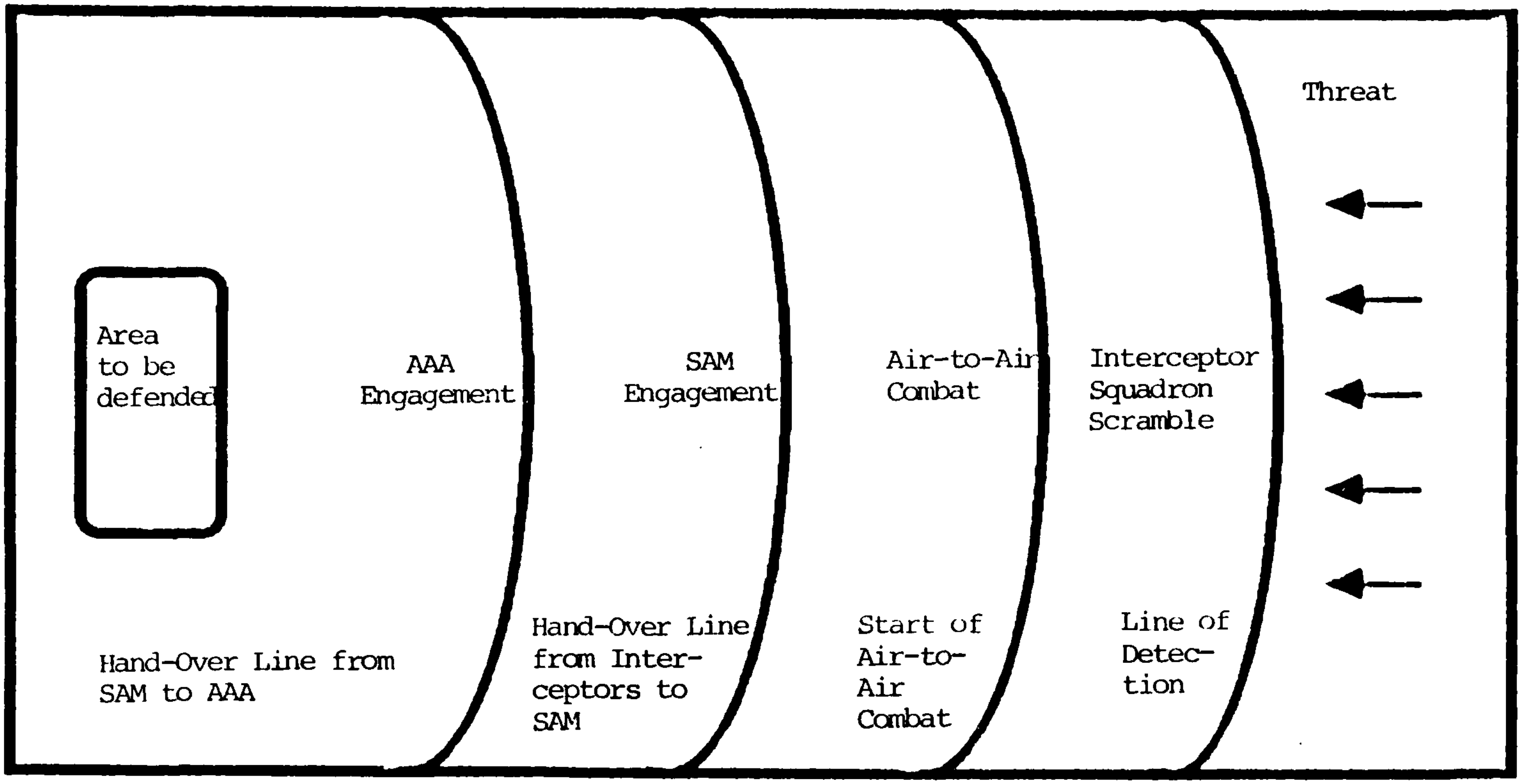

Figure 4.1. Stages of Air-Defense Combat

This study presents various models for the air-to-air combat process; these models will facilitate the linking of the operational logistics models presented in the previous chapters to produce the desired combat-logistics models. The combat-logistics models allow the decision maker to explicitly relate combat outcomes to logistics system scale and organization.

Some of the relevant measures-of-effectiveness (MOE's) for the air-interceptor squadron are the following:

1. The expected number of bombers reaching the first hand-over line; this measure is to be minimized. This step minimizes the expected number of bombers that will be engaged by the SAM units.

2. The probability that exactly z ($z \leq B(0)$), or no more than z, bombers leak through stage 1. Particular interest may well be in $z = 0$. The above measure is more general than the first-mentioned measure, but is also more difficult to compute. Diffusion approximations evaluate this measure in terms of the mean and variance of the leakage at hand-over time.

3. The duration of the air-to-air combat. Extending the combat duration will delay the bombers from reaching the first hand-over line. In turn, this will result in the bombers burning more fuel, which might result in bombers aborting their mission. This MOE in turn will also allow the SAM units longer warning time to prepare for combat. The manner in which air-to-air combat duration can be increased is by the range of first detection; which is a factor we study in the next chapter.

The general measure of the effectiveness of the air-interceptor squadron that will be considered (modelled and optimized) in this study is to minimize the probability of penetration of the enemy bombers. Here the term "penetration" refers to the ability of the enemy bombers to reach an imaginary line (hand-over line) beyond which lies an important

defense asset (e.g., SAM units), or beyond which the threat will release its load (i.e., bomb-release line).

Air-interceptor squadrons usually fight with a homogeneous set of aircraft. The squadron should be equipped so that it is capable of defeating an anticipated threat force. The opposing force structure at the beginning of a combat period ($t = 0$) is represented mathematically by the vector $(B(0), D(0))$, where:

$B(0)$ = The number of enemy Bombers at the beginning of the combat period.

$D(0)$ = The number of Defending aircraft at the beginning of the combat period.

Here the term "combat period" refers to the period during which the defenders and attackers (bombers) will meet in combat. It effectively begins when the bombers are in the range of the defenders. The duration of the combat period can be lengthened by extending the warning range. The study assumes that this duration is deterministic. It could be provided by the decision maker (e.g., points in time at which the decision maker is interested in knowing the opposing force distribution), or could be the time for the bombers to fly from point of detection/ intercept to hand-over line.

B. PROBLEM CHARACTERISTICS

The study considers two features of the combat situation: the early warning process, and the air-to-air combat process.

1. The Early Warning Process

The early warning process consists of the following operations.

a. Radar Detection of a Hostile Attack

Radar locates an aerial target in azimuth, range and altitude. It then estimates the target course, speed and strength (number of attacking bombers). The radar characteristics (particularly range) depend primarily upon the type and performance of the radar used, as well as the altitude selected for the antenna.

b. Target Identification

Identification enables targets to be classified in various categories (e.g., friendly, hostile, or unidentified). So a secondary radar which is identification--friendly or foe (IFF)--has to be used.

c. Threat Evaluation

This phase evaluates a threat index, which depends upon:

- The size of the attacking force.
- Geographical location.
- The required flight time to the high valued ground targets or installations.

d. Transmission of the Warning

Warning is transmitted to all air defense units (for example, interceptor squadron, SAM batteries, and the AAA units) and also to the areas being defended and even the civil population (shelters).

Successful execution of the above operations or missions is affected by factors such as:

- Radar range; the reliability and maintainability of the equipment, including adequacy of spare parts and maintenance facility, and equipment calibration.
- Training, skill, motivation, vigilance of the operators.
- Operational doctrine of the early warning system.
- The availability of, and coordination with, supplementary detection system.

Such supplementary systems include other friendly early warning systems in the area, as well as possible information from neighboring allied countries.

The objective of the early warning system assumed for this study is to maximize the time available for combat, by detecting the enemy threat far enough in advance to launch the air-interceptor squadron, so as to engage the incoming threat the longest possible time before reaching their bomb-release line. It is assumed that the longer is the available time for engagement (combat), the greater is the likely reduction in the incoming threat.

2. The Air-to-Air Combat Process

After detection of the incoming threat, and identified as hostile targets, the engagement of targets by the air-interceptor squadron requires the following steps.

a. Scrambling All Operational Aircraft

The scramble time is measured from announcement of the alert to the pilots until take-off. This time consists

of the time needed to start the aircraft engines, the time required to align the instruments, and the taxiing time.

b. Target Designation and Engagement

This step consists of assigning an enemy bomber, selected according to its threat index, to the aircraft most suited to achieve target interception and destruction. The outcome of the engagement is highly dependent on factors such as:

- Defender: Acceleration capability and tolerance, conditional kill probabilities given a hit, bomber silhouette dimensions, and the statistical nature of the projectile (missiles and/or guns) versus target range. Skill and training of the pilots also has an influence on the outcome of the engagement, together with the rate of fire that can be achieved by the weapon system.

- Bomber: Availability and usage of electronic countermeasure, accuracy of its defensive air-to-air weapon system, the amount of load carried, skill and motivation of pilots, and familiarity of pilots with defender's tactics.

- The reliability and effectiveness of the command and control system, and the skill and training of the ground operators.

C. MODELLING APPROACH

The complex problem of describing the interaction among the different variables, resulting from considering all of the preceding combat factors in detail, makes the use of a high-resolution combat simulation appear necessary for representing the combat process in detail, i.e., at a "micro" level. In this chapter we will represent the combat process alternatively and at a simple "macro" level by:

1. Deterministic (differential equations) models.
2. Diffusion processes.
3. Continuous-time-discrete-state Markov process.

Such representations, as will be seen, allow useful calculations to be made analytically. The models do not, however, recognize all detailed constraints to which the combat process is subject. Nevertheless, they, or their extensions, provide useful guidance to a decision maker.

The viewpoint taken to obtain macro-level combat models is familiar, see Taylor (1983). At any point of time during the combat process, a defender aircraft can be found to be either (i) engaging an enemy bomber, (ii) free and looking for a free bomber, or (iii) killed. Similarly, an enemy bomber can be found to be either engaged by a defender aircraft or free and trying to avoid being detected or found by defender aircraft, or killed. Suppose that we assume fight to the finish; that is, once a defender aircraft detects and engages an enemy bomber, the one-to-one engagement process will continue until either the bomber is killed or the defender aircraft is killed. (Note that only one free defender can engage a free bomber.) Under this scenario the combat process can be represented by the state vector $\{B(t), C(t), D(t);\ t \geq 0\}$ where:

$B(t)$ = The number of enemy Bombers free (and alive) at time t,

$C(t)$ = The number of enemy bombers (in turn defender aircraft) in Combat (i.e., engaged) at time t,

$D(t)$ = The number of Defender aircraft free (and alive) searching for a free bomber at time t.

Let α_D be the rate at which an individual defender in combat with a bomber kills that bomber (or causes a bomber to become ineffective); and α_B be the rate at which an individual bomber, in combat with a defender, kills that defender (or causes the defender to become ineffective).

Define θ to be the rate at which an individual free defender detects and starts engaging a free bomber, i.e., θ is the re-engagement rate. The units for these variables are given below:

$$\alpha_B \text{ is } \frac{\text{defenders killed}}{(\text{bombers})(\text{time})}$$

$$\alpha_D \text{ is } \frac{\text{bombers killed}}{(\text{defenders})(\text{time})}$$

$$\theta \text{ is } \frac{\text{bombers engaged}}{(\text{defenders})(\text{time})}$$

Note that the total number of bombers alive at time t is given by $B(t) + C(t)$, and the total number of defenders alive at time t is given by $D(t) + C(t)$.

Next, the simplest representations for the variables $B(t)$, $C(t)$, $D(t)$ will be presented: the process of mutual attrition is taken to be deterministic.

D. DETERMINISTIC MODELS

To simplify the representation of the combat process analysis, we introduce a deterministic representation of the

process by suppressing its randomness. Such a deterministic model often, as will be seen later, results in representations that deviate somewhat from the expectations of more sophisticated stochastic process models; such deviations may be referred to as the bias of the deterministic model. Usually the bias becomes small as $B(0)$ and $D(0)$ increase.

1. The Finite Re-Engagement Rate Combat Model; (the BCD Model)

First consider the situation described in the previous section in which θ, the re-engagement rate, is finite. The parameter θ presumably increases if the defense C^3 system is improved, but no attempt is made here to model C^3 in a detailed manner.

Define the following functions:

- $b(t)$, a representation of the number of bombers free and alive at time t, given the initial number of bombers and defenders entering combat initially.
- $c(t)$, a representation of the number of bombers and defenders engaged in combat at time t, given the initial number of bombers and defenders entering combat initially.
- $d(t)$, a representation of the number of defenders free and alive at time t, given the initial number of bombers and defenders entering combat initially.

RESULT (4.1):

A deterministic model for the above scenario is given by the following system of differential equations:

$$\frac{db(t)}{dt} = \alpha_B c(t) - \theta b(t)d(t)$$

$$\frac{dc(t)}{dt} = \theta b(t)d(t) - (\alpha_B+\alpha_D)c(t) \qquad (4.1)$$

$$\frac{dd(t)}{dt} = \alpha_D c(t) - \theta b(t)d(t)$$

with initial condition:

$$b(0) = B(0)$$

$$c(0) = C(0) = 0$$

$$d(0) = D(0)$$

The argument from which the equations is derived is as follows.

The first equation of (4.1) represents the rate of change of $b(t)$ with time. The term $\alpha_B c(t)$ represents the expected increase in $b(t)$ caused by a bomber killing a defender, while the term $\theta b(t)d(t)$ represents the decrease in $b(t)$ caused by a free defender detecting and engaging a free bomber.

The second equation of (4.1) represents the rate of change of $c(t)$ with time. The term $\theta b(t)d(t)$ is the expected increase in $c(t)$ caused by a free defender detecting and engaging a free bomber, while the term $(\alpha_B+\alpha_D)c(t)$ represents the decrease in $c(t)$ caused by a bomber killing a defender, or by a defender killing a bomber.

The third equation of (4.1) represents the rate of change of d(t) with time; the argument used for writing this equation is the same as that for the first equation.

Equations (4.1) can be solved numerically using Runge-Kutta method (Gerald, 1984) but not in simple explicit closed form. However, a limiting analysis can be as follows.

Adding the first equation of (4.1) to the second, and the third equation to the second of (4.1), we get:

$$\frac{d\{b(t)+c(t)\}}{dt} = -\alpha_D\, c(t)$$

$$\frac{d\{d(t)+c(t)\}}{dt} = -\alpha_B\, c(t) \qquad (4.2)$$

Define the following:

$\bar{b}(t)$, a representation of the total number of bombers alive at time t, and

$\bar{d}(t)$, a representation of the total number of defenders alive at time t.

Then:

$$\bar{b}(t) = b(t) + c(t)$$

$$\bar{d}(t) = d(t) + c(t) \qquad (4.3)$$

Substituting (4.3) into (4.2), we obtain:

$$\frac{d\bar{b}(t)}{dt} = - \alpha_D \, c(t)$$

$$\frac{d\bar{d}(t)}{dt} = - \alpha_B \, c(t) \tag{4.4}$$

From (4.4), we get:

$$\alpha_B \frac{d\bar{b}(t)}{dt} = \alpha_D \frac{d\bar{d}(t)}{dt} \tag{4.5}$$

Integrating both sides of (4.5), we obtain:

$$\alpha_B[\bar{b}(t) - \bar{b}(0)] = \alpha_D[\bar{d}(t) - \bar{d}(0)] \; , \quad \forall t \tag{4.6}$$

But from the initial conditions:

$$\bar{b}(0) = B(0) + C(0) = B(0)$$

and

$$\bar{d}(0) = D(0) + C(0) = D(0)$$

Therefore (4.6) becomes:

$$\alpha_B[\bar{b}(t) - B(0)] = \alpha_D[\bar{d}(t) - D(0)] \quad \forall t \tag{4.7}$$

Since α_B, α_D and $c(t)$ assume nonnegative values, then (4.4) indicates that $\bar{b}(t)$ and $\bar{d}(t)$ are both nonincreasing functions (as they should be).

a. Criterion for Defenders to Win, BCD Model

A criterion for the defenders to win can be obtained by noting that, in order to conclude that defenders won, it must be assumed that $\bar{b}(\infty) = 0$, and $\bar{d}(\infty) > 0$. Using this assumption, and letting $t \rightarrow \infty$, (4.7) becomes:

$$\alpha_D \bar{d}(\infty) = \alpha_D D(0) - \alpha_B B(0) \tag{4.8}$$

Since α_B, α_D, $\bar{d}(\infty)$ are all positive values, then:

$$\alpha_D D(0) - \alpha_B B(0) > 0$$

Therefore, for the assumption that defenders win to be true, we must have:

$$\alpha_D D(0) > \alpha_B B(0) \tag{4.9}$$

Hence, the condition for the defenders to win is expression (4.9). If the inequality is reversed, bombers win, i.e., some bombers survive.

Equation (4.8) also shows that:

$$\bar{d}(\infty) = \frac{\alpha_D D(0) - \alpha_B B(0)}{\alpha_D} \tag{4.10}$$

Therefore, the deterministic approximation for the number of defenders eventually surviving combat is

independent of the rate of engagement θ. This is not surprising, since time extends indefinitely. Of course an equation comparable to (4.10) predicts (approximately) the expected number of bombers surviving if (4.9) is reversed.

2. <u>The Infinite Re-Engagement Rate, Invulnerable Defender Model</u>

To simplify the representation of the combat process we can assume an Invulnerable defender's model (I scenario). Under the I scenario, we assume that the air defense system has a perfect C^3 system represented by having $\theta = \infty$, and that defenders are invulnerable, i.e., $\alpha_B = 0$, and only one-to-one engagement is allowed.

a. Deterministic Model for I Scenario

Under the I scenario assumptions, the combat process can be modelled deterministically by the following differential equation:

$$\frac{db(t)}{dt} = \begin{cases} -\alpha_D D(0) & \text{if } b(t) \geq D(0) \\ -\alpha_D b(t) & \text{if } b(t) < D(0) \end{cases} \qquad (4.11)$$

with initial condition:

$$b(0) = B(0)$$

Equation (4.11) represents the rate of change of b(t) with time. At any point in time if $b(t) \geq D(0)$ then all

defenders are in engagement (i.e., no free defenders), and there are $b(t) - D(0)$ free bombers. This is represented by the term $\alpha_D D(0)$. If $b(t) < D(0)$ then all bombers are being engaged (i.e., no free bombers), and there are $D(0) - b(t)$ free defenders, which justifies the term $-\alpha_D b(t)$.

The solution for equation (4.11) can be derived as follows:

Define:

$$t^* = \inf\{t: t \geq 0, b(t) < D(0)\}$$

Then, equation (4.11) becomes:

$$\frac{db(t)}{dt} = \begin{cases} -\alpha_D D(0) & \text{for } t \leq t^* \\ -\alpha_D b(t) & \text{for } t > t^* \end{cases} \tag{4.12}$$

where:

$$b(0) = B(0)$$

Note that if $B(0) \leq D(0)$, then $t^* = 0$; and the solution for (4.12) is given by:

$$b(t) = B(0)e^{-\alpha_D t} \quad \text{for } t \geq 0 \tag{4.13}$$

Suppose $B(0) > D(0)$. Then equation (4.12) says that there exists a time point, t^*, such that for any time point t less than t^*, $0 \leq t \leq t^*$, the rate of change of $b(t)$ is $-\alpha_D D(0)$. Recall that only one defender can engage a free bomber, so while bombers outnumber defenders, the rate of decrease is proportional to the defender numbers. After the point t^*, $b(t)$ declines below $D(0)$, where its rate of change becomes $-\alpha_D b(t)$.

For $t \leq t^*$, we have

$$\frac{db(t)}{dt} = -\alpha_D D(0)$$

Integrating both sides, and using the initial condition, we obtain:

$$b(t) = B(0) - \alpha_D D(0)t \quad \text{for } t \leq t^* \tag{4.14}$$

For $t > t^*$, we have:

$$\frac{db(t)}{dt} = -\alpha_D b(t)$$

Adding $\alpha_D b(t)$ to both sides, multiplying by $e^{\alpha_D t}$, and integrating, we obtain:

$$e^{\alpha_D t} b(t) - e^{\alpha_D t^*} b(t^*) = 0 \quad \text{for } t > t^* \tag{4.15}$$

From (4.14), we obtain:

$$b(t^*) = B(0) - \alpha_D D(0)t^* ,$$

substituting into (4.15) and multiplying by $e^{-\alpha_D t}$, we obtain:

$$b(t) = (B(0) - \alpha_D D(0)t^*)e^{-\alpha_D(t-t^*)} \quad \text{for } t > t^* \qquad (4.16)$$

Hence, from (4.14) and (4.16), it follows that:

$$b(t) = \begin{cases} B(0) - \alpha_D D(0)t & \text{for } t \leq t^* \\ (B(0) - \alpha_D D(0)t^*)e^{-\alpha_D(t-t^*)} & \text{for } t > t^* \end{cases} \qquad (4.17)$$

From the definition of t^*, we have:

$$b(t^*) \simeq D(0)$$

Substituting into (4.17) we get:

$$b(t^*) = B(0) - \alpha_D D(0)t^* \simeq D(0)$$

Hence,

$$t^* \simeq (B(0) - D(0))/\alpha_D D(0)$$

Equation (4.17) permits the easy direct computation of $b(t)$, the approximation for the number of bombers alive at various points of time during the combat period.

Note that in the preceding analysis we have assumed that the force levels are continuous variables. This is an assumption that we had to make in order to use the above differential equation models to develop deterministic representation for the evolution of the combat process. The tacit assumption of a deterministic process is perhaps even more questionable. Thus, we should develop stochastic analogues for the above deterministic models. A natural procedure is to utilize diffusion models; see Feller (1971).

In the development of our diffusion models we assume: (i) continuity of the force level variables, and (ii) Gaussian increments included in the process. We first consider a deterministic model, and then develop its stochastic analogue by representing the evolution of the combat system by a diffusion process, i.e., as if it evolves with independent Gaussian increments. Since the force levels $(B(t),D(t))$ are assumed to be continuous, the diffusion mathematics result in a mean and a covariance matrix that are approximations to the corresponding matrices that would occur if the process was modelled by a continuous-time, discrete-state Markov process. However, diffusions often yield good approximations with improvements occurring for large force level $(B(0) \to \infty,\ D(0) \to \infty)$. It follows that the

diffusion approximation can be used to approximate the MOE under study, which requires approximating the probability distribution of the number of free bombers at time t.

E. DIFFUSION MODELS FOR AIR-TO-AIR COMBAT

Following the arguments in Gaver and Lehoczky (1977a), we now formulate the combat processes resulting from modelling the previous scenarios by diffusion processes. Since we assume the continuity of the state variables, we will, therefore, adopt the following notation:

$\tilde{B}(t)$ a stochastic diffusion representation for the number of bombers free (and alive) at time t,

$\tilde{C}(t)$ a diffusion representation for the number of bombers (hence defenders) in engagement at time t, and

$\tilde{D}(t)$ a diffusion representation for the number of defenders free (and alive) at time t.

1. Diffusion Model for the BCD Scenario

Define N to be the total number of aircraft of both forces available for combat initially, i.e., $N \equiv B(0) + D(0)$. We aim to characterize the air-to-air combat process approximately when $N \to \infty$, by treating $\{\tilde{B}(t), \tilde{C}(t), \tilde{D}(t);\ t \geq 0\}$ as a trivariate diffusion process. When $N \to \infty$, we mean that $B(0) \to \infty$ and $D(0) \to \infty$ simultaneously and in a fixed proportion.

The behavior of the system state vector, $\{\tilde{B}(t), \tilde{C}(t), \tilde{D}(t);\ t \geq 0\}$, can be modelled directly by writing it in the form of an Ito stochastic differential equation as follows: Arnold (1974)

$$d\tilde{B}(t) = \alpha_B\tilde{C}(t)dt - \theta\tilde{B}(t)\tilde{D}(t)dt + \sqrt{\alpha_B\tilde{C}(t)}dW_1(t) - \sqrt{\theta\tilde{B}(t)\tilde{D}(t)}\,dW_2(t)$$

$$d\tilde{C}(t) = \theta\tilde{B}(t)\tilde{D}(t)dt - (\alpha_B+\alpha_D)\tilde{C}(t)dt - \sqrt{(\alpha_B+\alpha_D)\tilde{C}(t)}\,dW_1(t) + \sqrt{\theta\tilde{B}(t)\tilde{D}(t)}\,dW_2(t) \qquad (4.18)$$

$$d\tilde{D}(t) = \alpha_D\tilde{C}(t)dt - \theta\tilde{B}(t)\tilde{D}(t)dt - \sqrt{\alpha_D\tilde{C}(t)}\,dW_1(t) + \sqrt{\theta\tilde{B}(t)\tilde{D}(t)}\,dW_2(t)$$

where $\{W_1(t),W_2(t);\ t \geq 0\}$ is a bivariate standard Wiener process whose components are independent. That is, each component is normally distributed with zero mean and variance dt.

The rationale for writing (4.18) is as follows. The dt terms represent the drift or the expected behavior of the state variable, $[\tilde{B}(t),\tilde{C}(t),\tilde{D}(t)]$, between t and t+dt. As viewed by Gaver and Lehoczky (1977a), the evolution of the process can be considered as being that of a locally Poissonian process of rate $\alpha_B\tilde{C}(t)$, $\theta\tilde{B}(t)\tilde{D}(t)$ and $\alpha_D\tilde{C}(t)$ for $\tilde{B}(t)$, $\tilde{C}(t)$ and $\tilde{D}(t)$ respectively, with the average output rates being $\theta\tilde{B}(t)\tilde{D}(t)$, $(\alpha_B+\alpha_D)\tilde{C}(t)$ and $\theta\tilde{B}(t)\tilde{D}(t)$, respectively.

The term $\alpha_B\tilde{C}(t)$ in the first equation of (4.18) represents the expected increase in $\tilde{B}(t)$ caused by a bomber killing a defender, while the term of $\alpha_D\tilde{C}(t)$ in the third

equation represents the expected increase in $\tilde{D}(t)$ caused by a defender killing a bomber. The term $\theta\tilde{B}(t)\tilde{D}(t)$ used in equations (4.18) represents the decrease in $\tilde{B}(t)$ (hence $\tilde{D}(t)$), and the increase in $\tilde{C}(t)$, caused by a free defender detecting and engaging a free bomber. The variance of the Poisson equals the mean, and for large parameter values (in this case large number of initial bombers and defenders) the Poisson is approximately Gaussian; this heuristically justifies the coefficient of the Wiener process terms, and the Wiener terms themselves. An increase in $\tilde{B}(t)$ indicates a decrease in the total number of defenders alive $(\tilde{D}(t) + \tilde{C}(t))$; and vice versa. This is represented by the coefficients of the Wiener processes in the first equation of (4.18) having opposite signs than the corresponding terms in the third equation.

Consider the linear transformation:

$$
\begin{aligned}
\tilde{B}(t) &= N\tilde{b}(t) + \sqrt{N}\, X_1(t) \\
\tilde{C}(t) &= N\tilde{c}(t) + \sqrt{N}\, X_2(t) \\
\tilde{D}(t) &= N\tilde{d}(t) + \sqrt{N}\, X_3(t)
\end{aligned}
\tag{4.19}
$$

where: $\tilde{b}(t)$, $\tilde{c}(t)$, and $\tilde{d}(t)$ are deterministic functions of time, being approximations to the process means which need to be determined. $\{X_i(t);\ t \geq 0\}$, $i = 1,2,3$, are stochastic elements all of which need to be determined. Justification

of such transformations is given by Gaver and Lehoczky (1977a). Differentiating (4.19), we obtain:

$$\begin{aligned} d\tilde{B}(t) &= Nd\tilde{b}(t) + \sqrt{N}\, dX_1(t) \\ d\tilde{C}(t) &= Nd\tilde{c}(t) + \sqrt{N}\, dX_2(t) \\ d\tilde{D}(t) &= Nd\tilde{d}(t) + \sqrt{N}\, dX_3(t) \end{aligned} \tag{4.20}$$

Suppose $X_i(0) = 0$, $i = 1,2,3$. Then from (4.19), we obtain:

$$\begin{aligned} \tilde{b}(0) &= \frac{B(0)}{N} \\ \tilde{c}(0) &= 0 \\ \tilde{d}(0) &= \frac{D(0)}{N} \end{aligned} \tag{4.21}$$

Let $\tilde{\theta} = \theta N$ be constant as $N \to \infty$; i.e., assume that θ is small enough so that $\theta N \to$ constant as $N \to \infty$. This means that on the average a defender takes a relatively long time to detect and engage a bomber.

Now, substituting (4.19) and (4.20) into (4.18), and isolating terms of order N and $\sqrt{N}$ and letting $N \to \infty$, we obtain the following sets of equations.

a. Deterministic Equations

The terms of order N yield:

$$\frac{d\tilde{b}(t)}{dt} = \alpha_B\tilde{c}(t) - \tilde{\theta}\tilde{b}(t)\tilde{d}(t)$$

$$\frac{d\tilde{c}(t)}{dt} = \tilde{\theta}\tilde{b}(t)\tilde{d}(t) - (\alpha_B+\alpha_D)\tilde{c}(t) \tag{4.22}$$

$$\frac{d\tilde{d}(t)}{dt} = \alpha_D\tilde{c}(t) - \tilde{\theta}\tilde{b}(t)\tilde{d}(t)$$

with initial conditions given by (4.21).

The solution to these equations, which is obtained by numerical integration, provides a deterministic approximation to the means of the process. Note that equations (4.22) are equivalent to equations (4.1) where,

$$b(t) = N\tilde{b}(t)$$

$$c(t) = N\tilde{c}(t)$$

$$d(t) = N\tilde{d}(t)$$

This shows that the diffusion approximation results in the same approximations for the means as the deterministic model.

b. Stochastic Equations

The terms of order N yield:

$$
\begin{aligned}
dX_1(t) &= \{-\tilde{\theta}\tilde{d}(t)X_1(t)+\alpha_B X_2(t)-\tilde{\theta}\tilde{b}(t)X_3(t)\}dt \\
&\quad + \sqrt{\alpha_B\tilde{c}(t)}\, dW_1(t) - \sqrt{\tilde{\theta}\tilde{b}(t)\tilde{d}(t)}\, dW_2(t) \\
dX_2(t) &= \{\tilde{\theta}\tilde{d}(t)X_1(t) - (\alpha_B+\alpha_D)X_2(t) + \tilde{\theta}\tilde{b}(t)X_3(t)\}dt \\
&\quad - \sqrt{(\alpha_B+\alpha_D)\tilde{c}(t)}\, dW_1(t) + \sqrt{\tilde{\theta}\tilde{b}(t)\tilde{d}(t)}\, dW_2(t) \qquad (4.23) \\
dX_3(t) &= \{-\tilde{\theta}\tilde{d}(t)X_1(t)+\alpha_D X_2(t)-\tilde{\theta}\tilde{b}(t)X_3(t)\}dt \\
&\quad - \sqrt{\alpha_D\tilde{c}(t)}\, dW_1(t) + \sqrt{\tilde{\theta}\tilde{b}(t)\tilde{d}(t)}\, dW_2(t)
\end{aligned}
$$

To write (4.23) in a matrix form, let:

$$\vec{X}^T(t) = [X_1(t), X_2(t), X_3(t)] ,$$

and

$$\vec{W}^T(t) = [W_1(t), W_2(t)]$$

Then, (4.23) becomes:

$$d\vec{X}(t) = \begin{bmatrix} -\theta\tilde{\tilde{d}}(t) & \alpha_B & -\theta\tilde{\tilde{b}}(t) \\ \theta\tilde{\tilde{d}}(t) & -(\alpha_B+\alpha_D) & \theta\tilde{\tilde{b}}(t) \\ -\theta\tilde{\tilde{d}}(t) & \alpha_D & -\theta\tilde{\tilde{b}}(t) \end{bmatrix} \vec{X}(t)\, dt$$

$$+ \begin{bmatrix} \sqrt{\alpha_B \tilde{c}(t)} & -\sqrt{\theta\tilde{b}(t)\tilde{d}(t)} \\ -\sqrt{(\alpha_B+\alpha_D)\tilde{c}(t)} & \sqrt{\theta\tilde{b}(t)\tilde{d}(t)} \\ \sqrt{\alpha_D\tilde{c}(t)} & \sqrt{\theta\tilde{b}(t)\tilde{d}(t)} \end{bmatrix} d\vec{W}(t)$$

which appears in the form:

$$d\vec{X}(t) = A(t)\, \vec{X}(t)dt + V(t)\, d\vec{W}(t) \tag{4.24}$$

where $A(t)$ (a 3×3 matrix) being identified as the coefficient of $(X_1(t), X_2(t), X_3(t))^T$, and $V(t)$ (a 3×2 matrix) identified as the coefficient of the Wiener process term.

Now, since $\vec{X}(0) = \vec{0}$, and $\tilde{b}(0)$, $\tilde{c}(0)$ and $\tilde{d}(0)$ are given by (4.21), then by appealing to the central limit theorem, for all $t > 0$, $[X_1(t), X_2(t), X_3(t)]$ has a trivariate normal distribution with mean $\vec{0}$ and variance-covariance matrix $\Sigma(t)$ which satisfies the following differential equation; Arnold (1974):

$$\frac{d}{dt}\Sigma(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + V(t)V^T(t) \tag{4.25}$$

Recall that

$$\begin{bmatrix} \tilde{B}(t) \\ \tilde{C}(t) \\ \tilde{D}(t) \end{bmatrix} = N \begin{bmatrix} \tilde{b}(t) \\ \tilde{c}(t) \\ \tilde{d}(t) \end{bmatrix} + \sqrt{N} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}$$

Therefore, we have obtained the following:

<u>RESULT (4.2)</u>:

$\{\tilde{B}(t),\tilde{C}(t),\tilde{D}(t)\}$ is trivariate normal (Gaussian), as N (hence, $B(0) + D(0)) \to \infty$; where $B(0)$ and $D(0) \to \infty$ simultaneously and at a fixed proportion:

$$\{\tilde{B}(t),\tilde{C}(t),\tilde{D}(t)\} \simeq \text{Normal}(N(\tilde{b}(t),\tilde{c}(t),\tilde{d}(t)),N\Sigma(t))$$

From Result (2.1), it is possible to estimate the probability that at least a specified number of enemy bombers are free at time t, which is a useful measure of effectiveness (MOE).

The deterministic equations (4.22) and the equation (4.25) for the covariance matrix $\Sigma(t)$ can be solved by numerical integration.

By the required substitution and multiplication we get:

$$
V(t)V^T(t) = \begin{bmatrix} \alpha_B\tilde{c}(t)+\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & -\sqrt{\alpha_B(\alpha_B+\alpha_D)}\tilde{c}(t) -\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & -\sqrt{\alpha_B\alpha_D}\tilde{c}(t) -\theta\tilde{\tilde{b}}(t)\tilde{d}(t) \\ -\sqrt{\alpha_B(\alpha_B+\alpha_D)}\tilde{c}(t) -\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & (\alpha_B+\alpha_D)\tilde{c}(t) +\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & \sqrt{\alpha_D(\alpha_B+\alpha_D)}\tilde{c}(t) +\theta\tilde{\tilde{b}}(t)\tilde{d}(t) \\ -\sqrt{\alpha_B\alpha_D}\tilde{c}(t) -\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & \sqrt{\alpha_D(\alpha_B+\alpha_D)}\tilde{c}(t) +\theta\tilde{\tilde{b}}(t)\tilde{d}(t) & \alpha_D\tilde{c}(t) +\theta\tilde{\tilde{b}}(t)\tilde{d}(t) \end{bmatrix}
$$

Writing:

$$
\Sigma(t) = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \sigma_{13}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) & \sigma_{23}(t) \\ \sigma_{13}(t) & \sigma_{23}(t) & \sigma_{33}(t) \end{bmatrix}
$$

Equation (4.25) becomes:

$$
\frac{d}{dt}\vec{S}(t) = G(t)\,\vec{S}(t) + H(t) \tag{4.26}
$$

with initial conditions:

$$
\vec{S}(0) = \vec{0}
$$

where

$$
\vec{S}^T(t) = (\sigma_{11}(t)\ \sigma_{12}(t)\ \sigma_{13}(t)\ \sigma_{22}(t)\ \sigma_{23}(t)\ \sigma_{33}(t))
$$

G(t) is shown on the next page and H(t) is shown below.

$$H(t) = \begin{bmatrix} \alpha_B\tilde{c}(t)+\tilde{\theta}\tilde{b}(t)\tilde{d}(t) \\ -\sqrt{\alpha_B(\alpha_B+\alpha_D)}\tilde{c}(t)-\tilde{\theta}\tilde{b}(t)\tilde{d}(t) \\ -\sqrt{\alpha_B\alpha_D}\tilde{c}(t) - \tilde{\theta}\tilde{b}(t)\tilde{d}(t) \\ (\alpha_B+\alpha_D)\tilde{c}(t)+\tilde{\theta}\tilde{b}(t)\tilde{d}(t) \\ \sqrt{\alpha_D(\alpha_B+\alpha_D)}\tilde{c}(t)+\tilde{\theta}\tilde{b}(t)\tilde{d}(t) \\ \alpha_D\tilde{c}(t)+\tilde{\theta}\tilde{b}(t)\tilde{d}(t) \end{bmatrix}.$$

2. Diffusion Model for the I Scenario

Under the I scenario we can characterize the combat process approximately when $B(0) \to \infty$, and $D(0) \to \infty$, by treating $\{\tilde{B}(t);\ t \geq 0\}$ as a diffusion process.

The Ito stochastic differential equation is:

$$d\tilde{B}(t) = \begin{cases} -\alpha_D D(0)dt - \sqrt{\alpha_D D(0)}\, dW(t) & \text{for } \tilde{B}(t) \geq D(0) \\ -\alpha_D \tilde{B}(t)dt - \sqrt{\alpha_D \tilde{B}(t)}\, dW(t) & \text{for } \tilde{B}(t) < D(0) \end{cases} \qquad (4.27)$$

Equation (4.27) is written using the rationale used for BCD scenario. Let

$$D(0) = \delta B(0) \qquad (4.28)$$

where:

δ is fixed

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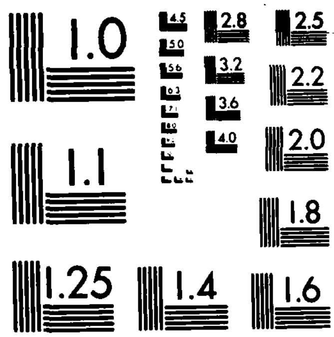

MICROCOPY RESOLUTION TEST CHART
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$$
G(t) = \begin{bmatrix}
-2\tilde{\theta}\tilde{d}(t) & 2\alpha_B & -2\tilde{\theta}\tilde{b}(t) & 0 & 0 & 0 \\
\tilde{\theta}\tilde{d}(t) & -(\tilde{\theta}\tilde{d}(t)+\alpha_B+\alpha_D) & \tilde{\theta}\tilde{b}(t) & \alpha_B & -\tilde{\theta}\tilde{b}(t) & 0 \\
-\tilde{\theta}\tilde{d}(t) & \alpha_D & -(\tilde{\theta}\tilde{d}(t)+\tilde{\theta}\tilde{b}(t)) & 0 & \alpha_B & -\tilde{\theta}\tilde{b}(t) \\
0 & 2\tilde{\theta}\tilde{d}(t) & 0 & -2(\alpha_B+\alpha_D) & 2\tilde{\theta}\tilde{b}(t) & 0 \\
0 & -\tilde{\theta}\tilde{d}(t) & \tilde{\theta}\tilde{d}(t) & \alpha_D & -(\tilde{\theta}\tilde{b}(t)+\alpha_B+\alpha_D) & \tilde{\theta}\tilde{b}(t) \\
0 & 0 & -2\tilde{\theta}\tilde{d}(t) & 0 & -2\alpha_D & -2\tilde{\theta}\tilde{b}(t)
\end{bmatrix}
$$

Consider the following linear transformation:

$$\tilde{B}(t) = B(0)\tilde{b}(t) + \sqrt{B(0)}\, X(t) \tag{4.29}$$

Differentiating (4.29), we obtain:

$$d\tilde{B}(t) = B(0)d\tilde{b}(t) + \sqrt{B(0)}\, dX(t) \tag{4.30}$$

Substituting (4.28)-(4.30) into (4.27), we obtain:

$$B(0)d\tilde{b}(t) + \sqrt{B(0)}dX(t) = \begin{cases} -\alpha_D \delta B(0)dt - \sqrt{\alpha_D \delta B(0)}\, dW(t) \\ \qquad \text{if } B(0)\tilde{b}(t)+\sqrt{B(0)}X(t) \geq D(0) = \delta B(0) \\ \\ -\alpha_D [B(0)\tilde{b}(t)+\sqrt{B(0)}X(t)]dt - \sqrt{\alpha_D [B(0)\tilde{b}(t)+\sqrt{B(0)}X(t)]}dW(t) \\ \qquad \text{if } B(0)\tilde{b}(t)+\sqrt{B(0)}X(t) < D(0) = \delta B(0) \end{cases} \tag{4.31}$$

Isolating terms of order $B(0)$ and $\sqrt{B(0)}$, and letting $B(0) \to \infty$, we obtain the following sets of equations.

a. Deterministic Equations

The terms of order $B(0)$ yield:

$$\frac{d\tilde{b}(t)}{dt} = \begin{cases} -\alpha_D \delta & \text{if } \tilde{b}(t) \geq \delta \\ \\ -\alpha_D \tilde{b}(t) & \text{if } \tilde{b}(t) < \delta \end{cases} \tag{4.32}$$

Let $t^* = \inf\{t: t > 0 \text{ and } b(t) < \delta\}$, from equation (4.32), it follows that:

$$\frac{d\tilde{b}(t)}{dt} = \begin{cases} -\alpha_D \delta & \text{if} \quad t \le t^* \\ -\alpha_D \tilde{b}(t) & \text{if} \quad t > t^* \end{cases} \tag{4.33}$$

Suppose $X(0) = 0$, then from equation (4.29), we obtain the following initial condition:

$$\tilde{b}(0) = 1 \tag{4.34}$$

For $t \le t^*$,

$$\frac{d\tilde{b}(t)}{dt} = -\alpha_D \delta,$$

therefore using (4.34), we obtain:

$$\tilde{b}(t) = 1 - \alpha_D \delta t \quad \text{for} \quad t \le t^* \tag{4.35}$$

For $t > t^*$, we have, from equation (4.33):

$$\frac{d\tilde{b}(t)}{dt} = -\alpha_D \tilde{b}(t)$$

$$\frac{d}{dt}\{e^{\alpha_D t}\tilde{b}(t)\} = 0$$

Hence:

$$e^{\alpha_D t}\tilde{b}(t) - \tilde{b}(t^*)e^{\alpha_D t^*} = 0 \quad \text{for} \quad t > t^* \tag{4.36}$$

From equation (4.48),

$$\tilde{b}(t^*) = 1 - \alpha_D \delta t^*$$

Substituting into equation (4.36), we get:

$$\tilde{b}(t) = \begin{cases} 1-\alpha_D\delta t & \text{for} \quad t \le t^* \\ (1-\alpha_D\delta t^*)e^{-\alpha_D(t-t^*)} & \text{for} \quad t > t^* \end{cases} \tag{4.37}$$

but from the definition of t*, we have:

$$\tilde{b}(t^*) \simeq \delta$$

substituting into (4.37), we get:

$$\tilde{b}(t^*) = 1 - \alpha_D\delta t^* \simeq \delta$$

Hence:

$$t^* \simeq \frac{1-\delta}{\alpha_D \delta} \tag{4.38}$$

b. Stochastic Equations

The terms of order $\sqrt{B(0)}$ yield:

$$dX(t) = \begin{cases} -\sqrt{\alpha_D\delta}\, dW(t) & \text{for } t \le t^* \\ -\alpha_D X(t)dt - \sqrt{\alpha_D \delta(t)}\, dW(t) & \text{for } t > t^* \end{cases} \tag{4.39}$$

For $t \le t^*$, from equation (4.39), we get:

$$dX(t) = -\sqrt{\alpha_D\delta}\, dW(t)$$

Integrating both sides, we get:

$$X(t) - X(0) = -\sqrt{\alpha_D\delta}[W(t) - W(0)] \tag{4.40}$$

Taking the expectation of both sides of equation (4.40), we obtain:

$$E[X(t)] = 0$$

Taking the variance of both sides of equation (4.40), we obtain:

$$Var(X(t)) = \alpha_D\delta t$$

Hence, for $t \leq t^*$, $X(t)$ is normally distributed with zero mean and variance $\alpha_D \delta t$.

For $t > t^*$, from equations (4.37) and (4.39), we get:

$$dX(t) = -\alpha_D X(t)dt - \sqrt{\alpha_D(1-\alpha_D\delta t^*)e^{-\alpha_D(t-t^*)}}\ dW(t)$$

$$\frac{d}{dt}\{e^{\alpha_D t}X(t)\} = -e^{\alpha_D t}\sqrt{\alpha_D(1-\alpha_D\delta t^*)e^{-\alpha_D(t-t^*)}}\ dW(t)$$

$$= -\sqrt{\alpha_D(1-\alpha_D\delta t^*)}\ e^{\frac{\alpha_D}{2}(t+t^*)}\ dW(t)$$

Integrating both sides we get:

$$e^{\alpha_D t}X(t)-e^{\alpha_D t^*}X(t^*) = -\sqrt{\alpha_D(1-\alpha_D\delta t^*)}\ e^{\frac{\alpha_D}{2}t^*} \times \int_{t^*}^{t} e^{\frac{\alpha_D}{2}\tau}\, dW(\tau)$$

Thus,

$$X(t) = e^{-\alpha_D(t-t^*)}X(t^*) - \sqrt{\alpha_D(1-\alpha_D\delta t^*)}\ e^{-\alpha_D(t-\frac{t^*}{2})} \times \int_{t^*}^{t} e^{\frac{\alpha_D}{2}\tau}\, dW(\tau) \qquad (4.41)$$

Since the function $e^{\frac{\alpha_D \tau}{2}}$ is a deterministic function then it is a nonanticipating function. The function g(s) is said to be a nonanticipating function if, for each s, g(s) is independent of $\{W(t) - W(s);\ t \geq s\}$; where W(t) is a Wiener process with mean zero, and variance t. That is, g(s) is independent of the future increments of the Wiener process after time s; Schuss (1980, p. 63). Hence, the integral

$$\int_{t^*}^{t} e^{\frac{\alpha_D \tau}{2}}\, dW(\tau)$$

is a stochastic integral (Arnold, 1974) and is a nonanticipating function (Schuss, 1980).

Now, by appealing to theorem (4.4.2) in Arnold (1974), we obtain:

$$E\left[\int_{t^*}^{t} e^{\frac{\alpha_D \tau}{2}}\, dW(\tau)\right] = 0 \tag{4.42}$$

and

$$E\left[\left\{\int_{t^*}^{t} e^{\frac{\alpha_D \tau}{2}}\, dW(\tau)\right\}^2\right] = \int_{t^*}^{t} e^{\alpha_D \tau}\, d\tau \tag{4.43}$$

It follows from equations (4.42) and (4.43) that:

$$\mathrm{Var}\{\int_{t^*}^{t} e^{\frac{\alpha_D \tau}{2}}\, dW(\tau)\} = \frac{1}{\alpha_D}\{e^{\alpha_D t} - e^{\alpha_D t^*}\} \tag{4.44}$$

Taking the expectation of both sides of equation (4.41) and using equation (4.42), we obtain:

$$E[X(t)] = e^{-\alpha_D (t-t^*)} E[X(t^*)]$$

But $X(t^*) \sim N(0,\alpha_D \delta t^*)$. Therefore we get

$$E[X(t)] = 0 \quad \text{for} \quad t > t^*.$$

Taking the variance of both sides of equation (4.41) and using equation (4.43), we obtain:

$$\mathrm{Var}(X(t)) = e^{-2\alpha_D (t-t^*)} \mathrm{Var}(X(t^*)) + \alpha_D (1-\alpha_D \delta t^*) e^{-2\alpha_D (t-\frac{t^*}{2})} \frac{1}{\alpha_D}[e^{\alpha_D t} - e^{\alpha_D t^*}]$$

$$= \alpha_D \delta t^* e^{-2\alpha_D (t-t^*)} + (1-\alpha_D \delta t^*)[e^{-\alpha_D (t-t^*)} - e^{-2\alpha_D (t-t^*)}]$$

Therefore, for $t > t^*$, we have:

$$X(t) \sim N\{0, \alpha_D \delta t^* e^{-2\alpha_D (t-t^*)} + (1-\alpha_D \delta t^*) e^{-\alpha_D (t-t^*)} (1-e^{-\alpha_D (t-t^*)})\}$$

We can summarize the above calculations as:

RESULT (4.3):

As $B(0) \to \infty$ (hence $D(0) \to \infty$), $\tilde{B}(t)$ is approximately normal (Gaussian).

$$\tilde{B}(t) \approx N(B(0)\tilde{b}(t), B(0)\mathrm{Var}(X(t))) .$$

This result can be easily used to compute leakage probabilities in terms of $D(0)$.

F. A MEASURE OF EFFECTIVENESS: AN APPROXIMATION

The diffusion approximation results in the same approximations for the process means as those given by the solutions of the corresponding deterministic models of the previous section. It has been shown above that, in addition to developing approximations for the process means, we are able to derive expressions for approximating the variance-covariance matrix of each process.

Note that the dimension of the system of differential equations resulting from the diffusion approximations is not a function of the initial size of the forces. For example, in the BCD scenario, we need to solve a system of 3 differential equations to solve for the deterministic means, and to solve a system of 6 differential equations for the variance-covariance matrix. This allows military analysts to economically model combat processes for larger size problems. It

also allows military OR analysts to derive expressions for approximating (or estimating) probability statements that represent measures of effectiveness (MOE), and to compute their values without expenditure of excessive computer time. That is, a useful MOE can be approximated analytically, and in near closed form; the results can be used in a program to optimize logistics allocations. For example, suppose that the MOE, as stated in the introduction to this chapter, is to minimize the probability of penetration of at least z free bombers. It is assumed that bombers are not capable of delivering their load, or their bombs will not hit targets defended, when they are being engaged by defenders. Define T_B to be the time required for the bombers to reach the bomb-release line (or the hand over line). Then $\tilde{B}(T_B)$ represents the number of bombers penetrating. The above MOE can be evaluated mathematically by:

$$P\{\tilde{B}(T_B) > z \mid B(0), D(0)\} \tag{4.45}$$

where the conditional distribution of $\{\tilde{B}(T_B) \mid B(0), D(0)\}$ given B(0), D(0) is given by Result (4.2) or Result (4.3), depending upon the scenario under consideration.

The diffusion assumes the continuity of the state variables; it is also an asymptotic approximation, i.e., it gives good approximation as B(0) and D(0) become large simultaneously. We can relax these assumptions, and evaluate their practical

validity, by formulating the combat process as a continuous-time discrete-state Markov process. This will require some other assumptions that will be stated in the next section.

G. MARKOVIAN COMBAT MODELS

Assume that the engagement time between a defender and a bomber is Markovian (i.e., has an exponential distribution). That is, a defender takes an exponential time with mean α_D^{-1} to kill a bomber, once an engagement started; and that a bomber takes an exponential time with mean α_B^{-1} to kill a defender. Assume, further, that the defender takes an exponential amount of time with mean θ^{-1} to detect and engage a free bomber. The air-to-air combat is formulated as a trivariate continuous-time-Markov process. Recall that we continue to assume that bombers are arriving simultaneously.

1. BCD Markovian Combat Model

Under the BCD scenario, the state of the process is represented by the trivariate-Markov process $\{B(t),C(t),D(t); t \geq 0\}$ operating on the state space S; where:

$$S = \{(i,j,k): 0 \leq i+j \leq B;\ 0 \leq j+k \leq D;\ 0 < i+j+k;$$
$$i = 0,1,\ldots,B;\ j = 0,1,\ldots,\min(B,D);\ k = 0,1,\ldots,D\}.$$

During a small interval of time $(t,t+dt]$, the state changes occur with the following probabilities:

t		$t+dt$	Probability
(i,j,k)	$\rightarrow$	$(i+1,j-1,k)$	$f_B(i,j,k)dt + o(dt)$
	$\rightarrow$	$(i-1,j+1,k-1)$	$f_C(i,j,k)dt + o(dt)$
	$\rightarrow$	$(i,j-1,k+1)$	$f_D(i,j,k)dt + o(dt)$
	$\rightarrow$	(i,j,k)	$1 - f(i,j,k)dt + o(dt)$

(4.45a)

where for the current model

$$
\begin{aligned}
f_B(i,j,k) &= \alpha_B j \\
f_C(i,j,k) &= \theta i j \\
f_D(i,j,k) &= \alpha_D j
\end{aligned}
\tag{4.45b}
$$

$$f(i,j,k) = f_B(i,j,k) + f_C(i,j,k) + f_D(i,j,k)$$

Let,

$$
\begin{aligned}
P_{mn\ell;ijk}(t) = P\{B(t) = i, C(t) = j, D(t) = k \mid B(0) = m, \\
C(0) = n, D(0) = \ell\}
\end{aligned}
$$

In what follows, for simplicity of notation, we will suppress the initial condition, and write:

$$P_{i,j,k}(t) = P_{mn\ell;ijk}(t)$$

Apparently $\{B(t), C(t), D(t);\ t \geq 0\}$ is a finite state space multivariate death process, as defined by (4.45).

The forward Chapman-Kolmogorov equations are derived by writing the probability that the process is in state (i,j,k) at time $t+dt$ as a function of the state probabilities at time t. The probability of being in state (i,j,k) at time $t+dt$ is, in general, expressed by:

$$
\begin{aligned}
P_{i,j,k}(t+dt) = {} & (1-f(i,j,k)dt)P_{i,j,k}(t) + f_C(i+1,j-1,k+1)dtP_{i+1,j-1,k+1}(t) \\
& + f_B(i-1,j+1,k)dtP_{i-1,j+1,k}(t) + f_D(i,j+1,k-1)dtP_{i,j+1,k-1}(t) \\
& + o(dt);
\end{aligned}
$$

the terms on the right-hand side express the mutually exclusive possibilities: (i) no state change, (ii) the number of combats increase by one, (iii) a bomber wins an engagement, (iv) a defender wins an engagement.

Subtracting $P_{i,j,k}(t)$ from both sides and dividing by dt, and letting dt tend to zero results in the following system of differential equations:

$$
\begin{aligned}
\frac{dP_{i,j,k}(t)}{dt} = {} & -f(i,j,k)P_{i,j,k}(t) + f_C(i+1,j-1,k+1)P_{i+1,j-1,k+1}(t) \\
& + f_B(i-1,j+1,k)P_{i-1,j+1,k}(t) + f_D(i,j+1,k-1)P_{i,j+1,k-1}(t) \ .
\end{aligned}
\tag{4.46}
$$

since $\frac{o(dt)}{dt} \to 0$ as $dt \to 0$. If $B(0) = B$, and $D(0) = D$, then the initial condition is:

$$P_{i,j,k}(0) = \begin{cases} 1 & \text{if } (i,j,k) = (B,0,D) \\ 0 & \text{otherwise} \end{cases}$$

Let

$$P'_{i,j,k}(t) = \frac{dP_{i,j,k}(t)}{dt}$$

The forward Chapman-Kolmogorov equations may be written out as follows, starting with the above initial conditions and recognizing boundary conditions explicitly:

$$P'_{B,0,D}(t) = -f(B,0,D)P_{B,0,D}(t)$$

$$P'_{B,0,k}(t) = -f(B,0,k)P_{B,0,k}(t) + f_B(B-1,1,k)P_{B-1,1,k}(t) \qquad 1 \leq k \leq D-1$$

$$P'_{i,0,D}(t) = -f(i,0,D)P_{i,0,D}(t) + f_D(i,1,D-1)P_{i,1,D-1}(t) \qquad 1 \leq i \leq B-1 \tag{4.47}$$

$$P'_{0,j,k}(t) = -f(0,j,k)P_{0,j,k}(t) + f_C(1,j-1,k+1)P_{1,j-1,k+1}(t)$$
$$+ f_D(0,j+1,k-1)P_{0,j+1,k-1}(t) \qquad k,j > 0 \quad 1 \leq k+j \leq D$$

$$P'_{i,j,0}(t) = -f(i,j,0)P_{i,j,0}(t) + f_C(i+1,j-1,1)P_{i+1,j-1,1}(t)$$

$$+ f_B(i-1,j+1,0)P_{i-1,j+1,0}(t) \quad i,j > 0 \quad 1 \leq i+j \leq B$$

$$P'_{i,j,k}(t) = -f(i,j,k)P_{i,j,k}(t) + f_C(i+1,j-1,k+1)P_{i+1,j-1,k+1}(t)$$

$$+ f_B(i-1,j+1,k)P_{i-1,j+1,k}(t) + f_D(i,j+1,k-1)P_{i,j+1,k-1}(t)$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$1 \leq i+j \leq B-1 \quad 1 \leq k+j \leq D-1$$

$$P'_{i,j,k}(t) = -f(i,j,k)P_{i,j,k}(t) + f_C(i+1,j-1,k+1)P_{i+1,j-1,k+1}(t)$$

$$+ f_B(i-1,j+1,k)P_{i-1,j+1,k}(t) \tag{4.47 (Cont'd)}$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$i+j = B \quad 1 \leq j+k \leq D-1$$

$$P'_{i,j,k}(t) = -f(i,j,k)P_{i,j,k}(t) + f_C(i+1,j-1,k+1)P_{i+1,j-1,k+1}(t)$$

$$+ f_D(i,j+1,k-1)P_{i,j=1,k-1}(t)$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$1 \leq i+j \leq B-1 \quad j+k = D$$

$$P'_{i,j,k}(t) = -f_C(i+1,j-1,k+1)P_{i+1,j-1,k+1}(t)$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$i+j = B \quad j+k = D$$

$$P'_{0,j,0}(t) = -f(0,j,0)P_{0,j,0}(t) + f_C(i,j-1,1)P_{1,j-1,1}(t)$$

$$1 \leq j \leq \min(B,D)$$

$$P'_{i,0,k}(t) = -f(i,0,k)P_{i,0,k}(t) + f_B(i-1,1,k)P_{i-1,1,k}(t) \qquad (4.47)\ (\text{Cont'd})$$

$$+ f_D(i,1,k-1)P_{i,1,k-1}(t)$$

$$1 \leq i \leq B-1 \qquad 1 \leq k \leq D-1$$

$$P'_{0,0,k}(t) = f_D(0,1,k-1)P_{0,1,k-1}(t) \qquad 1 \leq k \leq D$$

$$P'_{i,0,0}(t) = f_B(i-1,1,0)P_{i-1,1,0}(t) \qquad 1 \leq i \leq B$$

Let $\hat{P}_{i,j,k}(s)$ be the Laplace transform for $P_{i,j,k}(t)$, then:

$$\hat{P}_{i,j,k}(s) = \int_0^\infty e^{-st}P_{i,j,k}(t)dt \qquad s > 0$$

Taking the Laplace transform for $P_{B,0,D}(t)$, and using the initial condition $B(0) = B$, $C(0) = 0$, and $D(0) = D$, we get, from (4.45):

$$\int_0^\infty e^{-st}P'_{B,0,D}(t)dt = -f(B,0,D) \int_0^\infty e^{-st}P_{B,0,D}(t)dt$$

Using integration by parts, it follows that:

$$-1 + s\hat{P}_{B,0,D}(s) = -f(B,0,D)\hat{P}_{B,0,D}(s)$$

Therefore:

$$\hat{P}_{B,0,D}(s) = \frac{1}{s + f(B,0,D)}$$

Taking the Laplace transform for the general form of $P_{i,j,k}(t)$ in (4.47), we get:

$$\hat{P}_{i,j,k}(s) = \frac{1}{s + f(i,j,k)}\{f_C(i+1,j-1,k+1)\hat{P}_{i+1,j-1,k+1}(s)$$
$$+ f_B(i-1,j+1,k)\hat{P}_{i-1,j+1,k}(s)$$
$$+ f_D(i,j+1,k-1)\hat{P}_{i,j+1,k-1}(s)\} \, . \qquad (4.48)$$

Therefore, we obtain the following system of equations for the Laplace transforms:

$$\hat{P}_{B,0,D}(s) = \frac{1}{s + f(B,0,D)}$$

$$\hat{P}_{B,0,k}(s) = \frac{1}{s+f(B,0,k)}\{f_B(B-1,1,k)\hat{P}_{B-1,1,k}(s)\} \quad 1 \leq k \leq D-1 \qquad (4.49)$$

$$\hat{P}_{i,0,D}(s) = \frac{1}{s+f(i,0,D)}\{f_D(i,1,D-1)\hat{P}_{i,1,D-1}(s)\} \quad 1 \leq i \leq B-1$$

$$\hat{P}_{0,j,k}(s) = \frac{1}{s+f(0,j,k)}\{f_C(1,j-1,k+1)\hat{P}_{i,j-1,k+1}(s)$$

$$+ f_D(0,j+1,k-1)\hat{P}_{0,j+1,k-1}(s)\}$$

$$k,j > 0 \quad 1 \leq k+j \leq D$$

$$\hat{P}'_{i,j,0}(s) = \{\frac{1}{s+f(i,j,0)}\{f_C(i+1,j-1,1)\hat{P}_{i+1,j-1,1}(s)$$

$$+ f_B(i-1,j+1,0)\hat{P}_{i-1,j+1,0}(s)\}$$

$$i,j > 0 \quad 1 \leq k+j \leq D$$

$$\hat{P}_{i,j,k}(s) = \frac{1}{s+f(i,j,k)}\{f_C(i+1,j-1,k+1)\hat{P}_{i+1,j-1,k+1}(s) \qquad \text{(4.49) (Cont'd)}$$

$$+ f_B(i-1,j+1,k)\hat{P}_{i-1,j+1,k}(s)$$

$$+ f_D(i,j+1,k-1)\hat{P}_{i,j+1,k-1}(s)\}$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$1 \leq i+j \leq B-1 \quad 1 \leq k+j \leq D-1$$

$$\hat{P}_{i,j,k}(s) = \frac{1}{s+f(i,j,k)}\{f_C(i+1,j-1,k+1)\hat{P}_{i+1,j-1,k+1}(s)$$

$$+ f_B(i-1,j+1,k)\hat{P}_{i-1,j+1,k}(s)\}$$

$$1 \leq i \leq B-1 \quad 1 \leq k \leq D-1$$

$$i+j = B \quad 1 \leq j+k \leq D-1$$

$$\hat{P}_{i,j,k}(s) = \frac{1}{s+f(i,j,k)}\{f_C(i+1,j-1,k+1)\hat{P}_{i+1,j-1,k+1}(s)$$

$$+ f_D(i,j+1,k-1)\hat{P}_{i,j+1,k-1}(s)\}$$

$$1 \le i \le B-1 \quad 1 \le k \le D-1$$

$$j+k = D \quad 1 \le i+j \le B-1$$

$$\hat{P}_{i,j,k}(s) = \frac{1}{s+f(i,j,k)}\{f_C(i+1,j-1,k+1)\hat{P}_{i+1,j-1,k+1}(s)\}$$

$$1 \le i \le B-1 \quad 1 \le k \le D-1$$

$$i+j = B \quad j+k = D \tag{4.49 (Cont'd)}$$

$$\hat{P}_{0,j,0}(s) = \frac{1}{s+f(0,j,0)}\{f_C(1,j-1,1)\hat{P}_{1,j-1,1}(s)\} \quad 1 \le j \le \min(B,D)$$

$$\hat{P}_{i,0,k}(s) = \frac{1}{s+f(i,0,k)}\{f_B(i-1,1,k)\hat{P}_{i-1,1,k}(s)$$

$$+ f_D(i,1,k-1)\hat{P}_{i,1,k-1}(s)\} \quad 1 \le i \le B-1 \quad 1 \le k \le D-1$$

$$\hat{P}_{0,0,k}(s) = \frac{1}{s}\{f_D(0,1,k-1)\hat{P}_{0,1,k-1}(s)\} \quad 1 \le k \le D$$

$$\hat{P}_{i,0,0}(s) = \frac{1}{s}\{f_B(i-1,1,0)\hat{P}_{i-1,1,0}(s)\} \quad 1 \le i \le B$$

Equations (4.49) are in a form that is suitable for a recursive solution. Solving (4.49) recursively, we get:

$$\hat{P}_{B,0,D}(s) = \frac{1}{s+f(B,0,D)}$$

$$\hat{P}_{B-1,1,D-1}(s) = \frac{1}{s+f(B-1,1,D-1)}\{f_C(B,0,d)\hat{P}_{B,0,D}(s)\}$$

$$\hat{P}_{B,0,D-1}(s) = \frac{1}{s+f(B,0,D-1)}\{f_B(B-1,1,D-1)\hat{P}_{B-1,1,D-1}(s)\}$$

$$\hat{P}_{B-2,2,D-2}(s) = \frac{1}{s+f(B-2,2,D-2)}\{f_C(B-1,1,D-1)\hat{P}_{B-1,1,D-1}(s)\}$$

$$\hat{P}_{B-1,0,D}(s) = \frac{1}{s+f(B-1,0,D)}\{f_D(B-1,1,D-1)\hat{P}_{B-1,1,D-1}(s)\}$$

$$\hat{P}_{B-1,1,D-2}(s) = \frac{1}{s+f(B-1,1,D-2)}\{f_C(B,0,D-1)\hat{P}_{B,0,D-1}(s) + f_B(B-2,2,D-2)\hat{P}_{B-2,2,D-2}(s)\} \qquad (4.50)$$

$$\hat{P}_{B-3,3,D-3}(s) = \frac{1}{s+f(B-3,3,D-3)}\{f_C(B-2,2,D-2)\hat{P}_{B-2,2,D-2}(s)\}$$

$$\hat{P}_{B-2,1,D-1}(s) = \frac{1}{s+f(B-2,1,D-1)}\{f_C(B-1,0,D)\hat{P}_{B-1,0,D}(s) + f_D(B-2,2,D-2)\hat{P}_{B-2,2,D-2}(s)\}$$

$$\hat{P}_{B,0,D-2}(s) = \frac{1}{s+f(B,0,D-2)}\{f_B(B-1,1,D-2)\hat{P}_{B-1,1,D-2}(s)\}$$

$$\hat{P}_{B-2,2,D-3}(s) = \frac{1}{s+f(B-2,2,D-3)}\{f_C(B-1,1,D-2)\hat{P}_{B-1,1,D-2}(s) + f_B(B-3,3,D-3)\hat{P}_{B-3,3,D-3}(s)\}$$

$$\hat{P}_{B-1,0,D-1}(s) = \frac{1}{s+f(B-1,0,D-1)}\{f_B(B-2,1,D-1)\hat{P}_{B-2,1,D-1}(s) + f_D(B-1,1,D-2)\hat{P}_{B-1,1,D-2}(s)\}$$

$$\hat{P}_{B-4,4,D-4}(s) = \frac{1}{s+f(B-4,4,D-4)}\{f_C(B-3,3,D-3)\hat{P}_{B-3,3,D-3}(s)\}$$

$$\hat{P}_{B-3,2,D-2}(s) = \frac{1}{s+f(B-3,2,D-2)}\{f_C(B-2,1,D-1)\hat{P}_{B-2,1,D-1}(s) + f_D(B-3,3,D-3)\hat{P}_{B-3,3,D-3}(s)\} \tag{4.50 (Cont'd)}$$

$$\hat{P}_{B-2,0,D}(s) = \frac{1}{s+f(B-2,0,D)}\{f_D(B-2,1,D-1)\hat{P}_{B-2,1,D-1}(s)\} .$$

Continuing in this manner we can solve for the Laplace transforms recursively starting from state (B,0,D).

The state space, S, of the trivariate Markov process is a nonempty set of states. Define the set of edges E to be:

$$E = \{[(i,j,k),(n,m,\ell)]: (i,j,k) \text{ and } (n,m,\ell) \in S, \text{ and } (n,m,\ell) \text{ is accessible from } (i,j,k)\}$$

Since $\{B(t),C(t),D(t);\ t \geq 0\}$ is a multivariate pure death process, then E is a nonempty set of edges. The edges are ordered pairs $[(i,j,k),(n,m,\ell)]$ of states. The state (i,j,k)

is called the tail and the state (n,m,ℓ) is called the head of the edge $[(i,j,k),(n,m,\ell)]$. Let $G = (S,E)$. Then G is a directed graph. For a definition and discussion of graphs see Aho, Hopcroft and Ullman (1974). Figure (4.2) illustrates the graph G. Define the depth of a state (i,j,k) in G to be the length of the path from state $(B,0,D)$ to state (i,j,k), where the length of each edge is 1.

The set of equations (4.50) shows that by knowing the Laplace transforms of the states at the n^{th} depth of the graph, then the Laplace transforms for the $(n+1)^{st}$ depth can be calculated for any value of s. With this observation we can solve (4.49) by the following algorithm:

1. Construct the graph of the process. For the n^{th} depth find the possible states and calculate its widtn, i.e., how many states are in the n^{th} depth. Continue until the width of the depth is zero.

2. For the n^{th} depth, if width $(n) \neq 0$ then pick a state (say (i,j,k)). Then:

   2.1 Calculate the mean of the corresponding sojourn time. $((f(i,j,k))^{-1})$

   2.2 Examine all states in the $(n-1)^{st}$ depth, to determine which states are accessible to state (i,j,k).

   2.3 Calculate the corresponding transition rates from the state determined in (2.2) to (i,j,k).

   2.4 Calculate the Laplace transform for (i,j,k). Repeat for all states in the n^{th} depth. If width $(n) = 0$, stop.

Recall, in Chapter III we derived a one-to-one mapping between the multivariate Markov-process to a univariate Markov process. This mapping has been used for the above algorithm

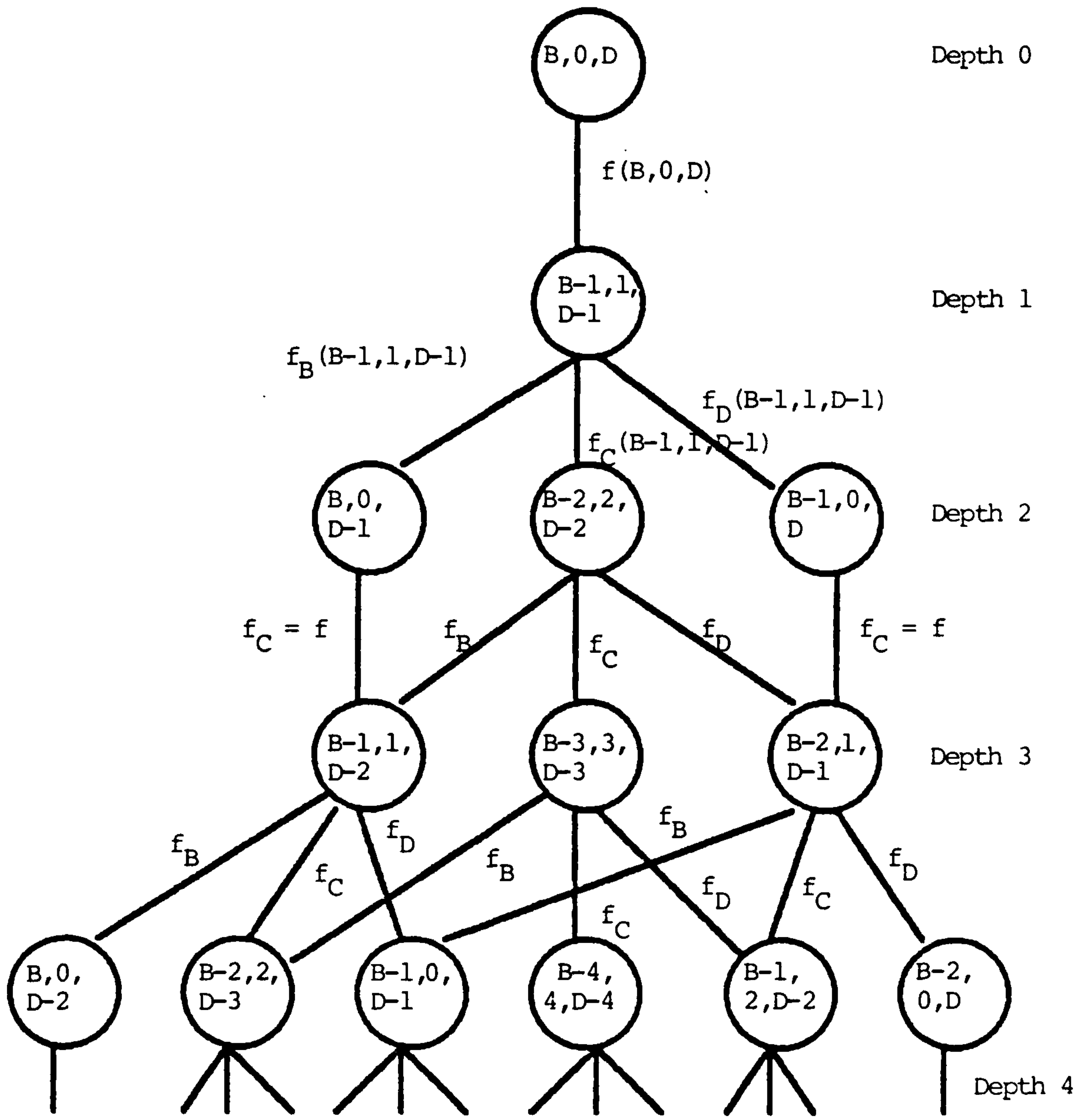


Figure 4.2. A Graph Representation of the BCD Combat Process

to be able to store and retrieve the states and their corresponding Laplace transforms.

The solutions for equations (4.50), i.e., the Laplace transforms can be used to evaluate some measures of interest as follows.

Suppose that the combat duration, T_B, is assumed to be a random variable distributed exponentially with mean ξ^{-1}. Let,

$$P_{i,j,k}(T_B) = P\{B(T_B) = i, C(T_B) = j, D(T_B) = k \mid B(0) = B, C(0) = 0, D(0) = D, T_B\}$$

Following the argument provided for evaluating the readiness of the squadron at the time of attack under the partial surprise scenario, we can express the combat outcome in terms of a Laplace transform, as follows:

$$E[P_{i,j,k}(T_B)] = \int_0^\infty P_{i,j,k}(u)\,\xi e^{-\xi u}\,du$$

$$= \xi\hat{P}_{i,j,k}(\xi) \qquad (4.51)$$

The probabilities $P_{i,j,k}(t)$ can be determined by inverting the Laplace transforms $\hat{P}_{i,j,k}(s)$; see Gaver (1966). A measure of interest is the probability of eventual survivors.

Let:

$$\pi_i(B) = P\{\text{i bombers eventually survive the combat}\}$$

From (4.47) we get:

$$P'_{i,0,0}(t) = f_B(i-1,1,0)P_{i-1,1,0}(t)$$

Integrating both sides, we obtain:

$$\int_0^t P'_{i,0,0}(\tau)d\tau = f_B(i-1,1,0) \int_0^t P_{i-1,1,0}(\tau)d\tau$$

Therefore:

$$P_{i,0,0}(t) = f_B(i-1,1,0) \int_0^t P_{i-1,1,0}(\tau)d\tau$$

$$\pi_i(B) = P_{i,0,0}(\infty)$$

$$= f_B(i-1,1,0) \int_0^\infty P_{i-1,1,0}(\tau)d\tau$$

Hence, in terms of Laplace transforms,

$$\pi_i(B) = f_B(i-1,1,0)\hat{P}_{i-1,1,0}(0) \qquad (4.52)$$

It follows that:

$$E[B(\infty)] = \sum_{i=1}^{B(0)} i\pi_i(B)$$

$$= \sum_{i=1}^{B(0)} i f_B(i-1,1,0)\hat{P}_{i-1,1,0}(0) \qquad (4.53)$$

Equations equivalent to (4.52) and (4.53) can be derived for the defenders by following similar arguments.

3. Markovian Combat Model for I Scenario

Under the I scenario, the state of the process is given by $\{B(t);\ t \geq 0\}$. The transition probabilities the process $\{B(t);\ t \geq 0\}$ of order dt are as follows:

$$\begin{array}{cccc} \underline{t} & & \underline{t+dt} & \underline{\text{Probability}} \\ i & \rightarrow & i-1 & f_D(i)dt + o(dt) \\ & \rightarrow & i & 1-f_D(i)dt + o(dt) \end{array} \qquad (4.54)$$

where in this particular case

$$f_D(i) = \alpha_D \min\{i, D(0)\} ; \qquad (4.55)$$

Actually, the remaining results hold quite generally for arbitrary $f_D(i)$.

Let $D \equiv D(0)$, and $B \equiv B(0)$. The stochastic process $\{B(t);\ t \geq 0\}$ defined by (4.54) is a pure death process with

state space $\{0,1,\ldots,B\}$; see e.g., Feller (1971), Chapter 14. The parameters of the process are given by:

$$f_D(i) = \alpha_D \min\{i,D\} = \begin{cases} i\alpha_D & \text{for } i = 0,1,2,\ldots,D-1 \\ D\alpha_D & \text{for } i = D,\ldots,B \end{cases} \tag{4.56}$$

Let:

$$P_{i,j}(t) = P\{B(t) = j \mid B(0) = i\} \tag{4.57}$$

For simplicity of notation, we will suppress the initial condition for the moment, and write:

$$P_j(t) \equiv P_{i,j}(t)$$

The forward Chapman-Kolmogorov equations can be derived, as in the BCD case; i.e., the probability of being in state (j) at time t+dt is expressed by:

$$P_j(t+dt) = (1-f_D(j)dt)P_j(t) + f_D(j+1)dtP_{j+1}(t) + o(dt)$$

It follows that,

$$P_j'(t) = -f_D(j)P_j(t) + f_D(j+1)P_{j+1}(t)$$

Therefore, the forward Chapman Kolmogorov equations are:

$$P_0'(t) = f_D(1)P_1(t)$$

$$P_j'(t) = -f_D(j)P_j(t) + f_D(j+1)P_{j+1}(t) \quad 1 \le j \le B-1 \qquad (4.58)$$

$$P_B'(t) = -f_D(B)P_B(t)$$

with initial condition:

$$P_j(0) = \begin{cases} 1 & \text{if } j = B \\ 0 & \text{otherwise} \end{cases}$$

Let $\hat{P}_j(s)$ be the Laplace transform for $P_j(t)$. Then:

$$\hat{P}_j(s) = \int_0^\infty e^{-st}P_j(t)dt \qquad s > 0$$

Taking the Laplace transforms of (4.58), using integration by parts and applying the initial condition, we obtain:

$$\hat{P}_B(s) = \frac{1}{f_D(B)+s}$$

$$\hat{P}_j(s) = \frac{1}{f_D(j)+s}\{f_D(j+1)\hat{P}_{j+1}(s)\} \quad 1 \le j \le B(0)-1 \qquad (4.59)$$

$$\hat{P}_0(s) = \frac{1}{s}\{f_D(1)\hat{P}_1(s)\}$$

Notice that (4.59) is already in a recursive form. Starting from state B(0) and proceeding recursively, we get:

$$\hat{P}_B(s) = \frac{1}{f_D(B)+s}$$

$$\hat{P}_{B-1}(s) = \frac{1}{f_D(B-1)+s}\left(\frac{f_D(B)}{f_D(B)+s}\right)$$

$$\hat{P}_{B-2}(s) = \frac{1}{f_D(B-2)+s}\left(\frac{f_D(B-1)}{f_D(B-1)+s}\ \frac{f_D(B)}{f_D(B)+s}\right)$$

Continuing this way, we get:

$$\hat{P}_j(s) = \frac{1}{f_D(j)+s} \prod_{n=j+1}^{B} \frac{f_D(n)}{f_D(n)+s}$$

Therefore, the solution for (4.59) is as follows:

$$\hat{P}_B(s) = \frac{1}{f_D(B)+s}$$

$$\hat{P}_j(s) = \frac{1}{f_D(j)+s} \prod_{n=j+1}^{B} \frac{f_D(n)}{f_D(n)+s} \qquad 1 \leq j \leq B-1 \tag{4.60}$$

$$\hat{P}_0(s) = \frac{1}{s} \prod_{n=1}^{B} \frac{f_D(n)}{f_D(n)+s}$$

Note that the term $f_D(n)/(f_D(n)+s)$ refers to the Laplace transform of the time spent in state (n).

a. Evaluation of Measures of Effectiveness: I Model

A useful measure of effectiveness is to compute the probability of at least z bombers alive at time T_B (recall T_B is assumed to be deterministic) in terms of B(0) and in particular, D(0) = D. The latter is influenced by logistics considerations. The probability of interest is:

$$P\{B(T_B > z|B(0),D(0)\}$$

Note that:

$$P\{B(T_B > z|B(0) = B,D(0) = D\} = \sum_{j=z+1}^{B} P_j(T_B) \tag{4.61}$$

The probabilities $P_j(t)$ can be determined by inverting the Laplace transforms $\hat{P}_j(s)$, see Ross (1983b).

Suppose T_B is assumed to be exponentially distributed with mean ξ^{-1}. Then as in the case of BCD, we can express the probability of interest in terms of the Laplace transforms; i.e., it can be shown that:

$$E[P\{B(T_B > z|B(0) = B,D(0) = D,T_B\}] = \xi \sum_{j=z+1}^{B} \hat{P}_j(\xi) \tag{4.62}$$

If we continue using the assumption that bombers being engaged by defenders are considered to be ineffective, then a measure of interest is to compute the probability of at least one free bomber reaching the bomb release line. Hence the probability of interest is: $P\{B(T_B) > D(0) | B(0), D(0)\}$. To evaluate this probability we construct an equivalence relationship with the sojourn times of the process in every state.

Let X_j, $j = 1,2,\ldots,B(0)$, be the sojourn times for $\{B(t); t \geq 0\}$ in state j. Therefore, X_j, $j = 1,\ldots,B$, are independent random variables, each with an exponential distribution $F_{X_j}(x) = 1 - e^{-f_D(j)x}$ that has a rate $D\alpha_D$ for $j = D, D=1,\ldots,B$, and a rate $j\alpha_D$ for $j = 1,2,\ldots,D-1$.

The event $\{B(T_B) > D(0) | B(0) = B, D(0) = D\}$ is equivalent to the event that the B(t) process at time T_B is at state j such that $j \in \{D+1, D+2,\ldots,B\}$.

Let S_D be the waiting time until there are exactly D bombers alive. At this point each bomber is being attacked by a defender (according to the present model). Then

$$S_D = \sum_{j=D+1}^{B} X_j \tag{4.63}$$

Thus, the event $\{B(T_B) > D(0) | B(0) = B, D(0) = D\}$ is equivalent to the event $\{S_D > T_B | B(0) = B, D(0) = D\}$. But S_D is a sum of i.i.d. exponential random variables with rate

$D\alpha_D$. Therefore, the conditional distribution of S_D given $B(0) = B$, and $D(0) = D$, is Gamma $(B-D, D\alpha_D)$.

Hence:

$$P\{B(T_B) > D \mid B(0) = B, D(0) = D\} = P\{S_D > T_B \mid B(0) = B, D(0) = D\}$$

$$= \int_{T_B}^{\infty} \frac{(D\alpha_D)^{B-D}}{(B-D-1)!} t^{B-D-1} e^{-D\alpha_D t} \, dt$$

Therefore, we obtain:

$$P\{B(T_B) > D \mid B(0) = B, D(0) = D\} = \sum_{n=1}^{B-D} \frac{(D\alpha_D t)^{n-1}}{(n-1)!} e^{-D\alpha_D t} \quad t \geq 0 \qquad (4.64)$$

Another measure of interest is to compute the expected number of bombers alive at time t, i.e., $E[B(t) \mid B(0), D(0)]$. To simplify the notation, we will temporarily suppress the initial condition, and write:

$$E[B(t)] = E[B(t) \mid B(0) = B, D(0) = D]$$

From the transition probabilities of $B(t)$ of order dt given by (4.54), we get:

$$B(t+dt) = \begin{cases} B(t) & \text{with probability } 1-f_D(B(t))dt \\ B(t)-1 & \text{with probability } f_D(B(t))dt \end{cases}$$

$$E[B(t+dt)|B(t)] \quad = \quad B(t) - f_D(B(t))dt$$

Taking expectation of both sides, we get,

$$E[B(t+dt)] \quad = \quad E[B(t)] - E[f_D(B(t))]dt$$

Subtracting E[B(t)] from both sides, dividing by dt and letting dt tend to zero, gives:

$$E'[B(t)] \quad = \quad -E[f_D(B(t))]$$

$$= \quad - \sum_{j=0}^{B} f_D(j)P_j(t)$$

From (4.55), it follows that:

$$E'[B(t)] \quad = \quad -\{\sum_{j=0}^{D} \alpha_D jP_j(t) + \sum_{j=D+1}^{B} \alpha_D DP_j(t)\} \tag{4.65}$$

Adding and subtracting $\sum_{j=D+1}^{B} \alpha_D jP_j(t)$ to the right-hand side of (4.65), we obtain:

$$E'[B(t)] \quad = \quad -\alpha_D E[B(t)] + \alpha_D \sum_{j=D+1}^{B} (j-D)P_j(t) \tag{4.66}$$

From (4.60), we have,

$$\hat{P}_j(s) = \frac{1}{f_D(j)+s} \prod_{n=j+1}^{B} \frac{f_D(n)}{f_D(n)+s} \qquad D+1 \leq j < B-1$$

$$\hat{P}_j(s) = \frac{(D\alpha_D)^{B-j}}{(D\alpha_D+s)^{B-j+1}} \qquad D+1 \leq j \leq B-1 \qquad (4.67)$$

since

$$f_D(j) = D\alpha_D \text{ for } D \leq j \leq B.$$

Let $L^{-1}\{\hat{P}_j(s)\}$ denotes the inversion of $\hat{P}_j(s)$, then:

$$P_j(t) = L^{-1}\{\hat{P}_j(s)\}$$

$$= L^{-1}\left\{\frac{(D\alpha_D)^{B-j}}{(D\alpha_D+s)^{B-j+1}}\right\}$$

$$P_j(t) = \frac{(D\alpha_D)^{B-j}}{(B-j)!} t^{B-j} e^{-D\alpha_D t} \qquad D+1 \leq j \leq B$$

Substituting into (4.78), we obtain:

$$E'[B(t)] = -\alpha_D E[B(t)] + \alpha_D \sum_{j=D+1}^{B} (j-D) \frac{(D\alpha_D t)^{B-j}}{(B-j)!} e^{-D\alpha_D t}$$

Adding $\alpha_D E[B(t)]$ to both sides of the above equation and multiplying by $e^{\alpha_D t}$, we obtain:

$$\frac{d}{dt}\{e^{\alpha_D t} E[B(t)]\} = \alpha_D \sum_{j=D+1}^{B} (j-D) \frac{(D\alpha_D t)^{B-j}}{(B-j)!} e^{-(D-1)\alpha_D t}$$

Integrating both sides, we obtain:

$$e^{\alpha_D t} E[B(t)]-E[B(0)] = \alpha_D \sum_{j=D+1}^{B} \frac{(j-D)(D\alpha_D)^{B-j}}{B(-j)!} \int_0^t \tau^{B-n} e^{-(D-1)\alpha_D \tau} d\tau$$

$$e^{\alpha_D t} E[B(t)]-B = \sum_{i=D+1}^{B} \frac{(j-D)D^{B-j}}{(D-1)^{B-j+1}} \left\{1 - e^{-(D-1)\alpha_D t} \sum_{k=0}^{B-j} \frac{((D-1)\alpha_D t)^k}{k!}\right\}$$

By transforming indices, we obtain:

$$E[B(t)] = Be^{-\alpha_D t} + e^{-\alpha_D t} \sum_{j=1}^{B-D} \frac{j}{D-1}\left(\frac{D}{D-1}\right)^{B-D-j} \left\{1 - e^{-(D-1)\alpha_D t}\right.$$

$$\left. \times \left\{1 + \sum_{k=1}^{B-D-j} \frac{((D-1)\alpha_D t)^k}{k'}\right\}\right\} \qquad B > D \qquad (4.68)$$

Note that if $B \leq D$, we obtain:

$$E[B(t)] = Be^{-\alpha_D t}$$

When the combat process starts with $B(0) = B \leq D$, then there are no free bombers. Every bomber will be engaged by a defender.

Define

$$Y_i(t) = \begin{cases} 1 & \text{if the } i^{th} \text{ defender is engaging a bomber} \\ 0 & \text{otherwise} \end{cases}$$

Then, $Y_i(t)$, $i = 1,2,\ldots,B$, are i.i.d. Bernoulli random variables with probability $e^{-\alpha_D t}$. Therefore, $B(t) = \sum_{i=1}^{B} Y_i(t)$ is a binomial random variable parameter $(B, e^{-\alpha_D t})$. Hence,

$$E[B(t)|B(0) = B, D(0) = D] = Be^{-\alpha_D t} \quad \text{if} \quad B \leq D \qquad (4.69)$$

Note: The above result can also easily be found if the combat times of individual bombers vs. defenders are <u>arbitrary</u> but i.i.d., denoted Z:

$$E[B(t)|B(0) = B, D(0) = D] = BP\{Z > t\}$$

We summarize the above in the following result.

<u>RESULT (4.4)</u>:

The expected number of bombers alive at time t for the I model is given by:

$$E\{B(t)|B(0)=B,D(0)=D\} = \begin{cases} Be^{-\alpha_D t} + e^{-\alpha_D t}\displaystyle\sum_{j=1}^{B-D} \frac{j}{D-1}\left(\frac{D}{D-1}\right)^{B-D-j}\left\{1-e^{-(D-1)\alpha_D t} \times \left\{1 + \sum_{k=1}^{B-D-j} \frac{((D-1)\alpha_D t)^k}{k!}\right\}\right\} & \text{if } B > D \\ Be^{-\alpha_D t} & \text{if } B \leq D \end{cases}$$

H. THE LARGE-DEVIATIONS APPROXIMATION FOR THE I MODEL

The large deviations technique is based on the idea of displacing the distribution towards a value of interest, applying the central limit theorem to the displaced distribution, and re-displacing. The resulting approximation will better estimate the behavior of the distribution at the value of interest, particularly when it is in the tail (is of small probability). For an exposition and some useful theory, see Feller (1971) and Esscher (1932); for an application see Mazumdar and Gaver (1984).

As noted earlier the combat process is, in the case of the I model, a pure-death process that starts initially from state B. The pure death process $\{B(t);\ t \geq 0\}$ operates on the state space $\{0,1,2,\ldots,B(0)\}$. Let X_j, $j = 0,1,\ldots,B$, denote the sojourn times for $\{B(t);\ t \geq 0\}$, i.e., X_j measures the duration that the process spends in state j. Therefore, X_j, $j = 0,1,\ldots,B$, are independent random variables with an

exponential distribution $F_{X_j}(\cdot)$ that has a rate $D\alpha_D$ for $j = D, D+1, \ldots, B$, and a rate $j\alpha_D$ for $n = 0, 1, \ldots, D-1$.

The model is constructed to evaluate $P\{B(t) > z | B(0) = B, D(0) = D\}$. The event $\{B(t) > z | B(0) = B, D(0) = D\}$ is equivalent to the event that the process at time t is at state j such that $j \in \{z+1, z+2, \ldots, B\}$. Let S_z be the waiting time to get exactly z bombers surviving. Then we have:

$$S_z = \sum_{j=z+1}^{B} X_j \qquad 0 \le z < B \tag{4.70}$$

Thus the event $\{B(t) > z | B(0) = B, D(0) = D\}$ is equivalent to the event $\{S_z > t | B(0) = B, D(0) = D\}$. It follows that:

$$P\{B(t) > z \; B(0) = B, D(0) = D\} = P\{S_z > t | B(0) = B, D(0) = D\}$$

$$= P\{\sum_{j=z+1}^{B} X_j > t | B(0) = B, D(0) = D\}$$

Note that the case $B \le D$ is a special case of a more general case where $B > D$. Therefore, in what is to follow, we will assume that $B > D$.

Let:

$$\phi_{X_j}(s) = \int_{-\infty}^{\infty} e^{st} F_{X_j}\{dt\}, \tag{4.71}$$

assumed to exist for real $s > 0$, $s_0 < s < 0$. Since the random variables X_j for $j = D, D+1, \ldots, B$ are i.i.d. random variables, then they have the same moment generating function. Let $\phi_X(s)$ be such a function.

Let $\hat{F}_{S_z}(s)$ be the moment generating function of S_z. To simplify the notation let $F(x) = F_{S_z}(x)$ and $\hat{F}(s) = \hat{F}_{S_z}(s)$. We will also suppress the initial condition. Then

$$\hat{F}(s) = E[e^{sS_z}] = E[e^{s\sum_{j=z+1}^{B} X_j}]$$

Since the random variables X_j are independent, we get:

$$\hat{F}(s) = (\phi_X(s))^{B-D} \prod_{j=z+1}^{D-1} \phi_{X_j}(s) \tag{4.72}$$

Let V be the probability distribution associated with $F(\cdot)$, with mean μ and variance σ^2 where:

$$V\{dx\} = \frac{e^{sx}F\{dx\}}{\hat{F}(s)}, \quad s > 0 \tag{4.73}$$

Let $\hat{V}(\xi)$ be the moment generating function of V. Then:

$$\hat{V}(\xi) = \int_{-\infty}^{\infty} e^{\xi x} V\{dx\} = \int_{-\infty}^{\infty} \frac{e^{(s+\xi)x}F\{dx\}}{\hat{F}(s)}$$

Therefore:

$$\hat{V}(\xi) = \frac{\hat{F}(\xi+s)}{\hat{F}(s)} \qquad \xi > 0, \quad s > 0 \tag{4.74}$$

Substituting equation (4.72) into equation (4.74), we get:

$$\hat{V}(\xi) = \left(\frac{\phi_X(\xi+s)}{\phi_X(s)}\right)^{B-D} \prod_{j=z+1}^{D-1} \frac{\phi_{X_j}(\xi+s)}{\phi_{X_j}(s)} \qquad \begin{matrix} \xi > 0 \\ s > 0 \end{matrix} \tag{4.75}$$

Let $K(\xi) \equiv \ell n\ \hat{V}(\xi)$ be the corresponding cumulant generating function. Then,

$$K(\xi) = (B-D)(\ell n\ \phi_X(\xi+s) - \ell n\ \phi_X(s))$$

$$+ \sum_{j=z+1}^{D-1} (\ell n\ \phi_{X_j}(\xi+s) - \ell n\ \phi_{X_j}(s)) \tag{4.76}$$

Differentiating equation (4.76) with respect to ξ, and evaluating the resulting function at $\xi = 0$, we get the following expression for the mean $\mu(s)$.

$$\mu(s) = \frac{(B-D)\,\phi_X'(s)}{\phi_X(s)} + \sum_{j=z+1}^{D-1} \frac{\phi_{X_j}'(s)}{\phi_{X_j}(s)} \tag{4.77}$$

Differentiating equation (4.76) twice with respect to ξ, and evaluating at $\xi = 0$, we get the following expression for the variance:

$$\sigma^2(s) = \frac{(B-D)\{\phi_X(s)\phi_X''(s)-(\phi_X'(s))^2\}}{(\phi_X(s))^2} + \sum_{j=z+1}^{D-1} \frac{\phi_{X_j}(s)\phi_{X_j}''(s)-(\phi_{X_j}'(s))^2}{(\phi_{X_j}(s))^2} \qquad (4.78)$$

where:

$$\phi_X(s) = \frac{D\alpha_D}{D\alpha_D - s} \qquad s > 0 \qquad (4.79)$$

$$\phi_{X_j}(s) = \frac{j\alpha_D}{j\alpha_D - s} \qquad s > 0,\ j = 0,1,\ldots,D-1 \qquad (4.80)$$

$$\phi_X'(s) = \frac{D\alpha_D}{(D\alpha_D - s)^2} \qquad s > 0 \qquad (4.81)$$

$$\phi_{X_j}'(s) = \frac{j\alpha_D}{(j\alpha_D - s)^2} \qquad s > 0,\ j = 0,1,\ldots,D-1 \qquad (4.82)$$

$$\phi_X''(s) = \frac{2D\alpha_D}{(D\alpha_D - s)^3} \qquad s > 0 \qquad (4.83)$$

and

$$\phi_{X_j}''(s) = \frac{2j\alpha_D}{(j\alpha_D - s)^3} \qquad s > 0,\ j = 0,1,\ldots,D-1 \qquad (4.84)$$

Note that the mean and variance of the associated distribution, equations (4.77) and (4.78), respectively, are

functions of s. Therefore, there is a family of associated distributions for $F_{S_z}(\cdot)$. Let $\mu(s) = t$ (i.e., center the associated distribution at the point of interest), and solve for s. Let $\tilde{s}$ be the value of s such that $\mu(s) = t$. Thus, an associated distribution V has been determined. Therefore, equation (4.73) becomes:

$$V\{dx\} = \frac{e^{\tilde{s}x} F\{dx\}}{\hat{F}(\tilde{s})} \qquad s > 0 \tag{4.85}$$

with:

$$\text{mean} = \mu(\tilde{s}) = t$$

and

$$\text{variance} = \sigma^2(\tilde{s})$$

Thus:

$$F\{dx\} = \hat{F}(\tilde{s}) e^{-\tilde{s}x} V\{dx\} \tag{4.86}$$

and

$$P\{S_z > t\} = \int_t^\infty F\{dx\}$$

Substituting, we get:

$$P\{S_z > t\} = \hat{F}(\tilde{s}) \int_t^\infty e^{-\tilde{s}x} V\{dx\} \tag{4.87}$$

Invoke a normal approximation to V; this is justified if the sum belongs to the domain of attraction of the normal distribution according to the central limit theorem. Certainly the initial B-D components are i.i.d. and have a second moment by assumption, and hence the CLT does apply to their sum. Provided z is not too small (e.g., if $z = kB(0)$, k fixed), then the remaining terms are also well behaved, and the normal approximation should be adequate.

Therefore, it follows that

$$P\{S_z > t\} \simeq P_{LD}(S_z > t) \equiv \hat{F}(\tilde{s}) \frac{1}{\sqrt{2\pi}} \int_{\mu(\tilde{s})}^\infty e^{-\tilde{s}x} e^{\frac{-(x-\mu(\tilde{s}))^2}{2\sigma^2(\tilde{s})}} \frac{dx}{\sigma(\tilde{s})}$$

By standard substitution, $w = \frac{x-\mu(s)}{\sigma(\tilde{s})}$, we obtain:

$$P_{LD}\{S_z > t\} = \hat{F}(\tilde{s}) \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\tilde{s}(\mu(\tilde{s})+w\sigma(\tilde{s}))} e^{-\frac{w^2}{2}} dw$$

Multiplying the right-hand side by $e^{\tilde{s}^2\sigma^2(\tilde{s})-\tilde{s}^2\sigma^2(\tilde{s})}$, to complete the square, we get:

$$P_{LD}\{S_z > t\} = \hat{F}(\tilde{s}) e^{-\tilde{s}\mu(\tilde{s})+\frac{1}{2}\tilde{s}^2\sigma^2(\tilde{s})} \int_0^\infty e^{-\frac{1}{2}(w+\tilde{s}\sigma(\tilde{s}))^2} dw$$

Let $x = w + \tilde{s}\sigma(\tilde{s})$. From the above equation, we obtain:

$$P_{LD}\{S_z > t\} = \hat{F}(\tilde{s}) e^{-\tilde{s}\mu(\tilde{s}) + \frac{1}{2}\tilde{s}^2\sigma^2(\tilde{s})} \frac{1}{\sqrt{2\pi}} \int_{\tilde{s}\sigma(\tilde{s})}^{\infty} e^{-x^2/2}\,dx \qquad (4.88)$$

Using equations (4.72) and (4.79) to (4.82), we obtain:

$$\hat{F}(\tilde{s}) = \left(\frac{D\alpha_D}{D\alpha_D - \tilde{s}}\right)^{B-D} \left(\prod_{j=z+1}^{D-1} \frac{j\alpha_D}{j\alpha_D - \tilde{s}}\right)$$

Since $\mu(\tilde{s}) = t$, equation (4.88) becomes:

$$P_{LD}\{S_z > t\} = \left(\frac{D\alpha_D}{D\alpha_D - \tilde{s}}\right)^{B-D} \left(\prod_{j=z+1}^{D-1} \frac{j\alpha_D}{j\alpha_D - \tilde{s}}\right) e^{-\tilde{s}t + \frac{1}{2}\tilde{s}^2\sigma^2(\tilde{s})} (1 - \Phi(\tilde{s}\sigma(\tilde{s}))) \qquad (4.89)$$

where:

$\Phi(x)$ is the standard normal probability distribution function, and

$$\sigma^2(\tilde{s}) = \frac{B-D}{(D\alpha_D - \tilde{s})^2} + \sum_{j=z+1}^{D-1} \frac{1}{(j\alpha_D - \tilde{s})^2},$$

and $\tilde{s}$ is the appropriate solution of:

$$\frac{B-D}{D\alpha_D - \tilde{s}} + \sum_{j=z+1}^{D-1} \frac{1}{j\alpha_D - \tilde{s}} = t \qquad (4.90)$$

To solve for $\tilde{s}$, we note that $\tilde{s} > 0$ since it is the transform variable for the moment generating function. We also note that in order to have $P\{S_z > t\} \geq 0$, we must have $\frac{j\alpha_D}{j\alpha_D - \tilde{s}} > 0$ for every $j = z+1,\ldots,D-1$. Hence, $\tilde{s} < (z+1)\alpha_D$. Therefore we obtain the following condition on $\tilde{s}$:

$$0 < \tilde{s} < (z+1)\alpha_D \tag{4.91}$$

Define:

$$t^* = \frac{B-D}{D\alpha_D} + \sum_{j=z+1}^{D-1} \frac{1}{j\alpha_D}$$

Hence, if $t > t^*$, then $\tilde{s} > 0$. We now need to determine if for all values of $t > t^*$ there will always exist a $0 < \tilde{s} < (z+1)\alpha_D$ such that equation (4.90) is true.

Define:

$$f(s) \equiv \frac{B-D}{D\alpha_D - s} + \sum_{j=z+1}^{D-1} \frac{1}{j\alpha_D - s} - t$$

The function $f(s)$ is a piece-wise continuous (differentiable) function with points of discontinuity $s = j\alpha_D$, $j = z+1,\ldots,D$. Differentiating $f(s)$ over the ranges of continuity (differentiability), we get:

$$f'(s) = \frac{B-D}{(D\alpha_D - s)^2} + \sum_{j=z+1}^{D-1} \frac{1}{(j\alpha_D - s)^2} \quad \text{for } 0 < s < (z+1)\alpha_D$$

and for $j\alpha_D < s < (n+1)\alpha_D$; $j = z+2,\ldots,D$.

From the model assumption we have $B \geq D$. Therefore

$$f'(s) \geq 0 \quad \forall \quad 0 < s < (z+1)\alpha_D; \quad j\alpha_D < s < (j+1)\alpha_D$$

Hence, the function $f(s)$ is a piecewise continuous function, and each piece is a monotonically nondecreasing function over its corresponding range.

Note that $f(0) < 0$ for $t > t^*$, and $f((z+1)\alpha_D) = \infty$ for any t. Thus, by imposing the condition $t > t^*$, there will always exist $0 < \tilde{s} < (z+1)\alpha_D$ such that $f(\tilde{s}) = 0$. The bisection method (Gerald, 1984), was implemented to determine $\tilde{s}$ numerically.

Hence, the Large-Deviation approximation for the invulnerable combat model becomes:

$$P\{B(t) > z|B(0) = B, D(0) = D\} \simeq \left(\frac{D\alpha_D}{D\alpha_D - \tilde{s}}\right)^{B-D} \left(\prod_{j=z+1}^{D-1} \frac{j\alpha_D}{j\alpha_D - \tilde{s}}\right) e^{-\tilde{s}t + \frac{1}{2}\tilde{s}^2\sigma^2(\tilde{s})}$$

$$\times \quad (1 - \Phi(\tilde{s}\sigma(\tilde{s})))$$

where:

$$t > \frac{B-D}{D\alpha_D} + \sum_{j=z+1}^{D-1} \frac{1}{j\alpha_D} \qquad B > D$$

$$\sigma^2(\tilde{s}) = \frac{B-D}{(D\alpha_D - \tilde{s})^2} + \sum_{j=z+1}^{D-1} \frac{1}{(j\alpha_D - \tilde{s})^2}$$

and

$$0 < \tilde{s} < (z+1)\alpha_D$$

such that:

$$\frac{B-D}{D\alpha_D - \tilde{s}} + \sum_{j=z+1}^{D-1} \frac{1}{j\alpha_D - \tilde{s}} = t$$

In Appendix A we derive a general equation for the large deviations approximation for a fixed and a randomized sum of i.i.d. random variables, together with some applications to other logistics problems.

I. SIMULATION OF BCD MARKOVIAN MODEL

In order to check the multivariate diffusion approximations for the BCD model, a simulation was carried out by using Monte-Carlo methods to generate a realization of the air-to-air combat process, which is modelled as a trivariate-continuous-time discrete-state Markov process. Combat data were then generated to compute combat statistics.

The Monte-Carlo simulation was constructed based on the following:

a. The time spent by the process at state (i,j,k) is exponentially distributed with rate f(i,j,k). If desired, the exponential distribution can be replaced by an arbitrary distribution function to obtain a semi-Markov model.

Let $X_{i,j,k}$ be the sojourn time for state (i,j,k). Then:

$$P\{X_{i,j,k} \leq t\} = \begin{cases} 1 - e^{-f(i,j,k)t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad (4.92)$$

b. The transition probabilities for the embedded Markov chain are as follows:

$$\begin{array}{lclc} \underline{\text{Present}} & & \underline{\text{Next}} & \underline{\text{Probability}} \\ (i,j,k) & \rightarrow & (i+1,j-1,k) & \dfrac{f_B(i,j,k)}{f(i,j,k)} \\ & \rightarrow & (i-1,j+1,k-1) & \dfrac{f_C(i,j,k)}{f(i,j,k)} \\ & \rightarrow & (i,j-1,k+1) & \dfrac{f_D(i,j,k)}{f(i,j,k)} \end{array} \qquad (4.93)$$

The conditions in (a) and (b) are used, together with a uniform random number generator that produces independent samples of a random variable that is uniformly distributed over the interal (0,1), so as to generate a realization of the combat process. This is done by generating two independent samples of a unit uniform variate, denoted by U_1 and U_2.

U_1 is used to produce a realization of the time $X_{i,j,k}$, time spent in state (i,j,k), by applying the inverse transform method, as follows:

$$P\{X_{i,j,k} \leq x_{i,j,k}\} = 1 - e^{-f(i,j,k)x} = U$$

then

$$e^{-f(i,j,k)x_{i,k}} = 1 - U = U_1$$

Therefore,

$$-f(i,j,k)x_{i,j,k} = \ell n\ U_1$$

Thus, we obtain, as is well-known,

$$x_{i,j,k} = \frac{-\ \ell n\ U_1}{f(i,j,k)} \tag{4.94}$$

The number U_2 is used to determine which state the process will enter next. After spending $x_{i,j,k}$ units of time in state (i,j,k), the process will next transit to:

$$\begin{array}{lll} (i+1,j-1,k) & \text{if} & 0 < U_2 \leq \dfrac{f_B(i,j,k)}{f(i,j,k)} \\ (i-1,j+1,k-1) & \text{if} & \dfrac{f_B(i,j,k)}{f(i,j,k)} < U_2 \leq \dfrac{f_B(i,j,k) + f_C(i,j,k)}{f(i,j,k)} \\ (i,j-1,k+1) & & \text{otherwise} \end{array} \tag{4.95}$$

We collect statistics for the process by discretizing the time domain. Let Δt be the length of the time increments, and T be the length of time for the simulation of the process. T could be the time for bombers to be in region from detection to hand-over line. Define N to be the number of increments in T. Then:

$$N = \frac{T}{\Delta t}$$

Construct the following:

BB,BS to be arrays of size (N+1) whose entries are

$$BB(n) = \sum_{m=1}^{M} B_m(n\Delta t); \quad BS(n) = \sum_{m=1}^{M} B_m^2(n\Delta t)$$

CC,CS to be arrays of size (N+1) whose entries are

$$CC(n) = \sum_{m=1}^{M} C_m(n\Delta t); \quad CS(n) = \sum_{m=1}^{M} C_m^2(n\Delta t)$$

DD,DS to be arrays of size (N+1) whose entries are

$$DD(n) = \sum_{m=1}^{M} D_m(n\Delta t), \quad DS(n) = \sum_{m=1}^{M} D_m^2(n\Delta t)$$

where:

$$(B_m(n\Delta t), C_m(n\Delta t), D_m(n\Delta t))$$

is a (the m^{th}) realization of the state of the process at time $n\Delta t$ for the m^{th} replication, and M is the total number of replications.

Let $\bar{B}(t)$ and $S^2(B(t))$ be the sample average and variance, respectively, for B(t). Then:

$$\bar{B}(n\Delta t) = \frac{\sum_{m=1}^{M} B_m(n\Delta t)}{M} \qquad n = 0,1,\ldots,N \qquad (4.96)$$

and

$$S^2(B(n\Delta t)) = \sum_{m=1}^{M} \frac{(B_m(n\Delta t) - \bar{B}(n\Delta t))^2}{M-1} \qquad n = 0,1,\ldots,N \quad (4.97)$$

It can be shown that the above are unbiased estimators for the process mean and variance (Larson, 1983).

Equations similar to (4.96) and (4.97) can be constructed for C(t), and D(t).

J. NUMERICAL ILLUSTRATION FOR VARIOUS COMBAT MODELS

This section presents some numerical illustrations for the various combat models presented above. The illustrations will be conducted by comparing the diffusion approximations with the simulation estimations or the actual parameters

when possible. Therefore, for the BCD combat model we compare the diffusion approximations to the simulation estimations, while for the I combat models we compare the diffusion and the large deviations approximations to the actual values.

1. <u>An Illustration for the BCD Combat Model</u>

The following basic data were used as an example for model illustration:

- A threat of $B(0) = 12$ bombers attacking an area of responsibility of an interceptor squadron that is assigned exactly $a = 40$ aircraft. Therefore, at the time of attack, the interceptor squadron will have $D(0)$ aircraft ready to engage in combat where $0 \leq D(0) \leq a$.
- Each interceptor aircraft (and bomber) is assumed to take an exponential amount of time, to kill its opponent with mean $(\alpha_D)^{-1} = (.008)^{-1}$ seconds (and $(\alpha_B)^{-1} = (.004)^{-1}$ seconds) respectively.
- The command and control system is assumed to take an exponential amount of time, with mean $(\theta)^{-1} = (.006)^{-1}$ seconds, to detect and engage a free bomber.

FORTRAN programs were written for:

1. Computing the approximate mean and variance for $B(t)$ using the diffusion approximation.
2. Simulating the discrete-state Markovian combat process to compute the mean, and variance for $B(t)$. The number of replications $M = 10,000$.

For illustration purposes the following values for $D(0)$ were considered:

$$D(0) = 6, 12, 16, 20, 30, \text{ and } 40 .$$

Figure (4.3) shows plots of the simulation mean, and the deterministic mean resulting from the diffusion model

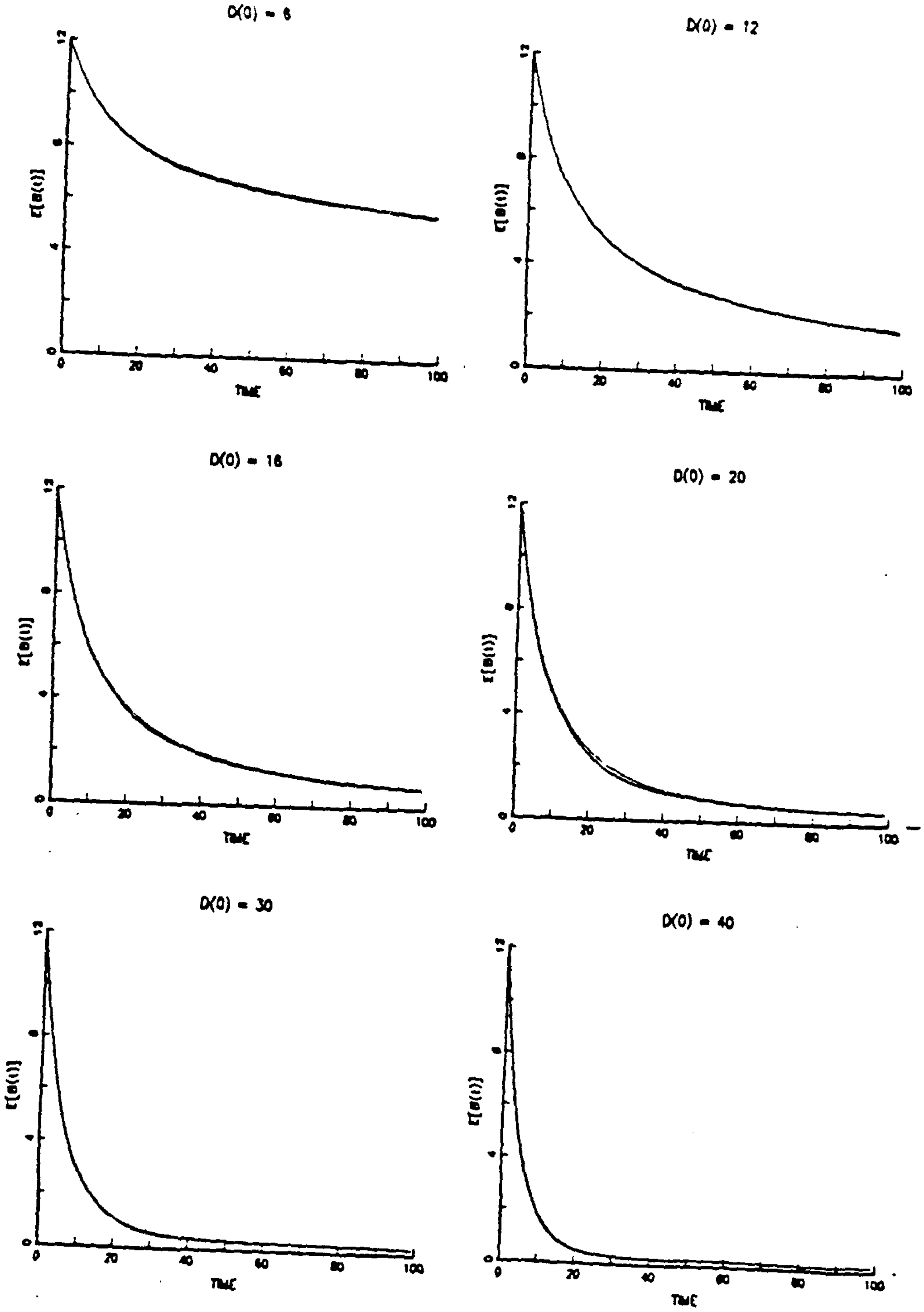


Figure 4.3. Plots of Deterministic Mean Versus the Stochastic Mean of B(t): BCD Scenario

for the above cases. It is clear from the plot that the diffusion approximation produces a good approximation for the mean of the process.

Figure (4.4) shows a comparison of the variance function resulting from the diffusion model with the variance function resulting from simulating the Markovian combat process, for the above different values of D(0). The figure shows that the diffusion approximation provides a good approximation to the variance as D(0) increases. Since we are assuming an increase of $B(0) + D(0) \to \infty$, where B(0) and $D(0) \to \infty$ simultaneously and at a fixed proportion. We therefore consider the following 3 cases ($k = B(0)/D(0) = 0.6$).

	B(0)	D(0)
Case 1:	12	20
Case 2:	15	25
Case 3:	18	30

Figure (4.5) shows a plot of the resulting variance functions of B(t) for the above cases as determined by the simulation model and the diffusion approximation. The figure shows that the variance approximation resulting from the diffusion model greatly improves as B(0) and $D(0) \to \infty$ simultaneously and at a fixed proportion. Note that the diffusion variance is systematically larger than the simulation variance, but the discrepancy is reduced as time advances. Since it is probably the relatively long

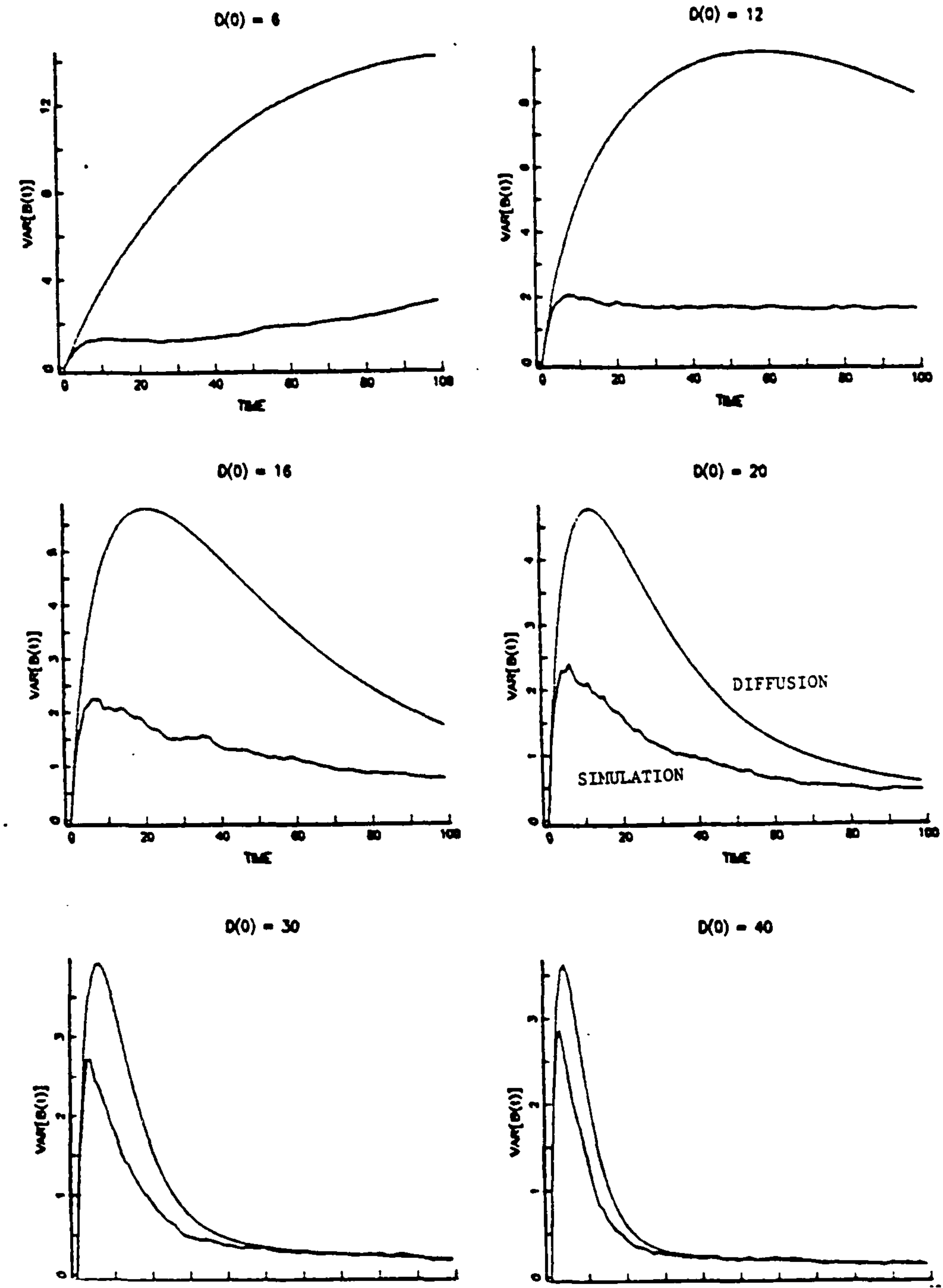


Figure 4.4. Plots of the Variance Approximation Resulting from Diffusion Versus Variance from Simulation, for B(t): BCD Scenario

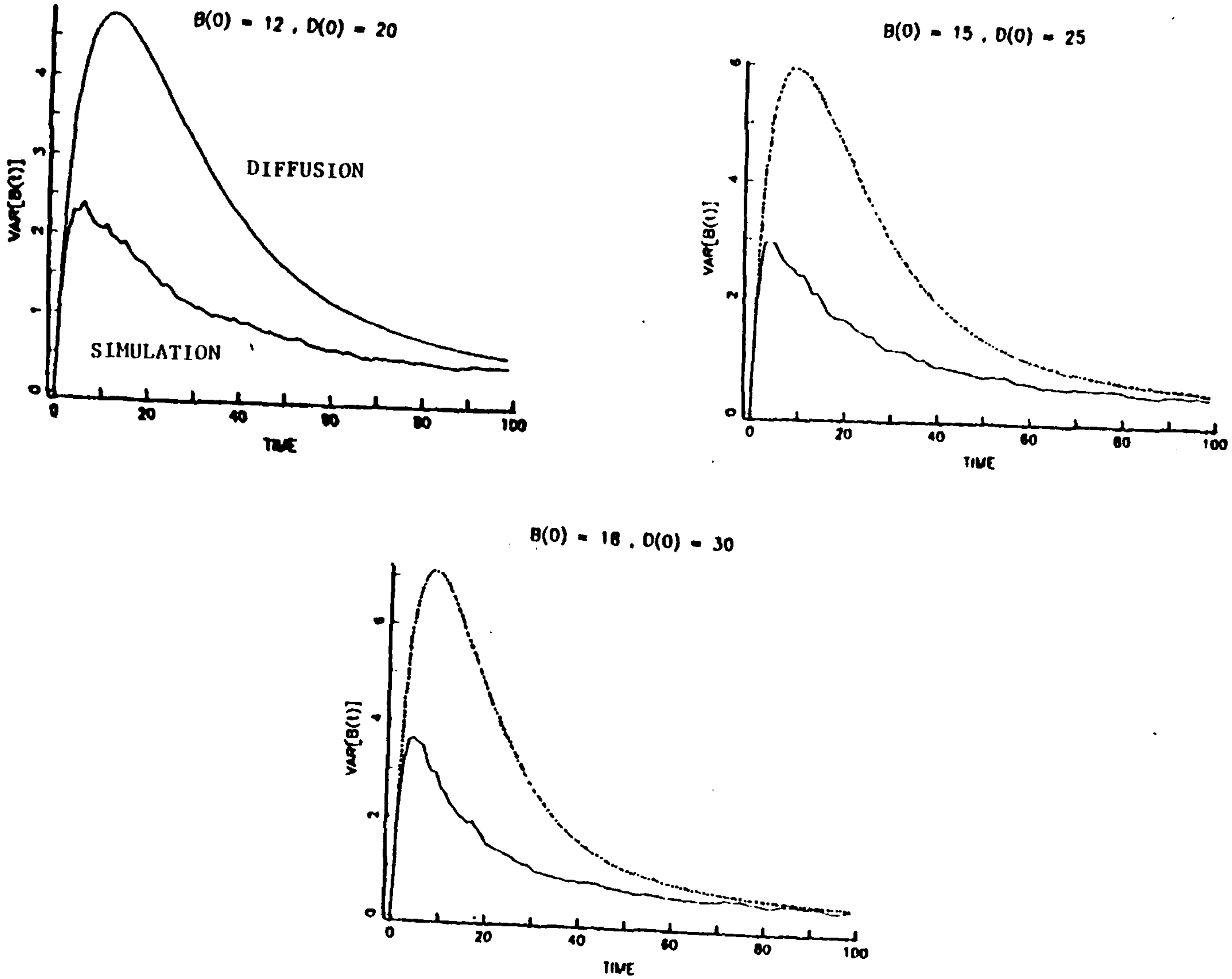


Figure 4.5. A comparison of the Diffusion and Simulation Variance of B(t) as Both B(0) and D(0) $\to \infty$ Simultaneously and in a Proportion of .6: BCD Scenario

times (of flight of the bombers towards their objective) that is of interest, the discrepancy may not be serious, but in any case the direction of the difference is conservative: The diffusion model will predict greater leakage of bombers than does the (presumed) more exact simulation. The diffusion computation is far more computationally economical. Its adoption actually permits many more rapid calculations than would use of simulation.

2. An Illustration for the Invulnerable Combat Model

The following basic data were used as an example for model illustration:

- A threat of $B(0) = 20$ bombers attacking an area of responsibility of an interceptor squadron.
- The squadron has exactly $D(0) = 13$ aircraft operational and ready to engage.
- Each interceptor aircraft is assumed to take an exponential amount of time to kill a bomber with mean $(\alpha_D)^{-1} = 20$ minutes.
- An interceptor aircraft is assumed to be invulnerable, and is controlled by a perfect command and control system. Therefore the time taken for a free defender to start engaging a free bomber is instantaneous, i.e., equal to zero.
- The intelligence has reported that the enemy tactical doctrine dictates that the raid to be cancelled if there are $z = 2$ bombers or less alive. Hence, we are interested in evaluting $P\{B(t) > 2 | B(0) = 20, D(0) = 13\}$.

FORTRAN programs were written for:

1. Computing the stochastic mean, stochastic variance and probabilities using the Markov-process formulation.
2. Computing the approximate mean, and variance using the diffusion approximation, and then applying the central limit theorem to compute the required probability.

3. Computing the approximate probability using the large-deviations approximations.

Figure (4.6) shows a plot for the stochastic mean compared to the deterministic mean. We find that the deterministic mean yields a good approximation for the stochastic mean. To further investigate that, a plot of the stochastic mean versus the deterministic mean, and a plot of the difference between the stochastic mean and the deterministic mean versus time, are shown in Figure (4.7). It is obvious from the figure that the two means form a 45° line when plotted against each other. Figure (4.7) shows that the bias between the 2 means tends to zero as $t \to \infty$. The bias is defined to be the difference between the stochastic mean and the deterministic mean.

$$\text{Bias} = \text{Stochastic mean} - \text{Deterministic mean}$$

Figure (4.7) also shows that the bias is always positive and tends to increase from zero at $t = 0$ to a maximum bias of .167, and then starts to decrease to zero as $t \to \infty$. This shows that the deterministic mean underestimates the stochastic mean for the initial stages of combat.

Figure (4.8) shows a plot for the exact variance compared to the variance resulting from the diffusion approximation. We find that the variance obtained by diffusion gives a good approximation to the exact variance. Plots of the

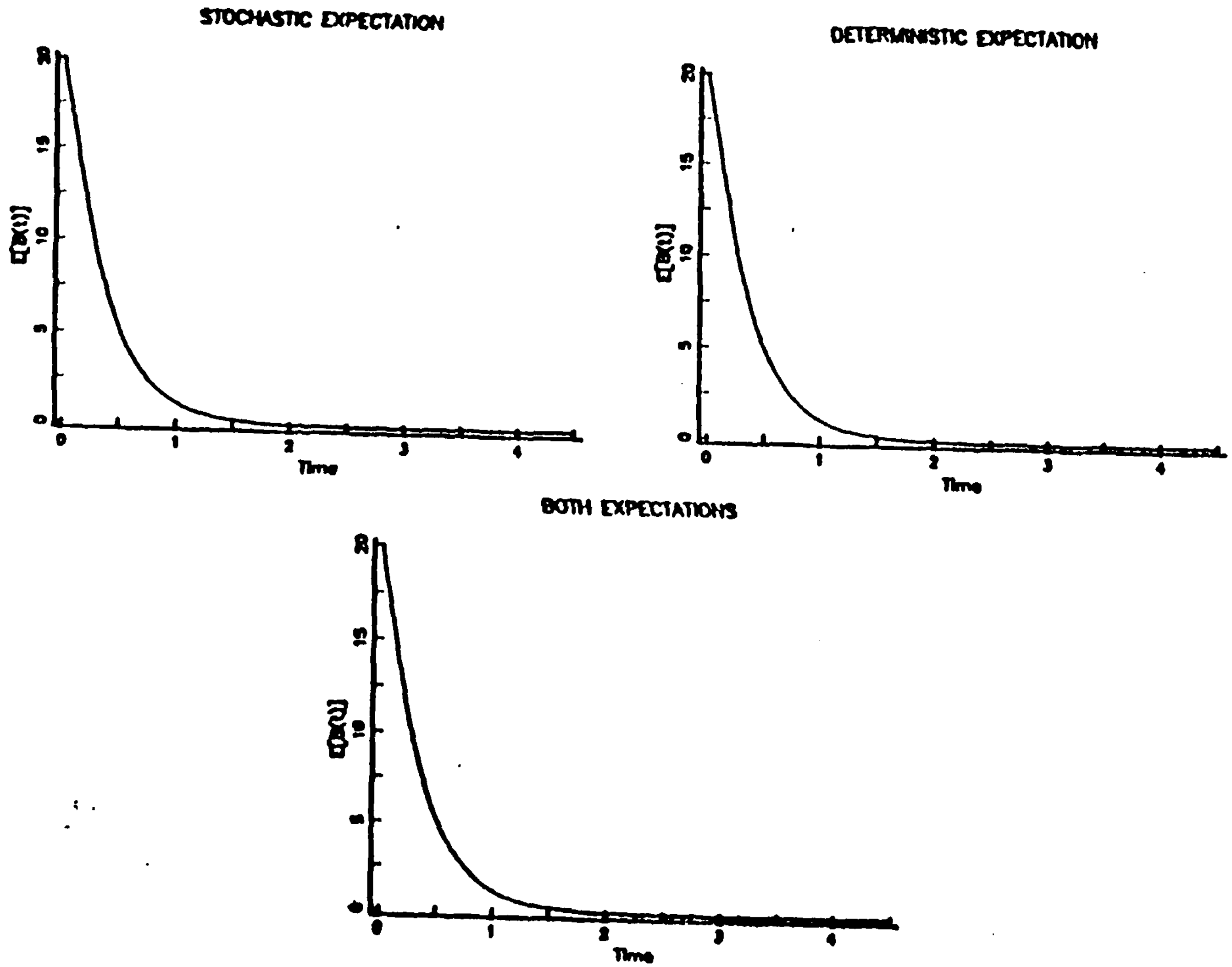


Figure 4.6. Stochastic Expectation Versus Deterministic Expectation; I Scenario

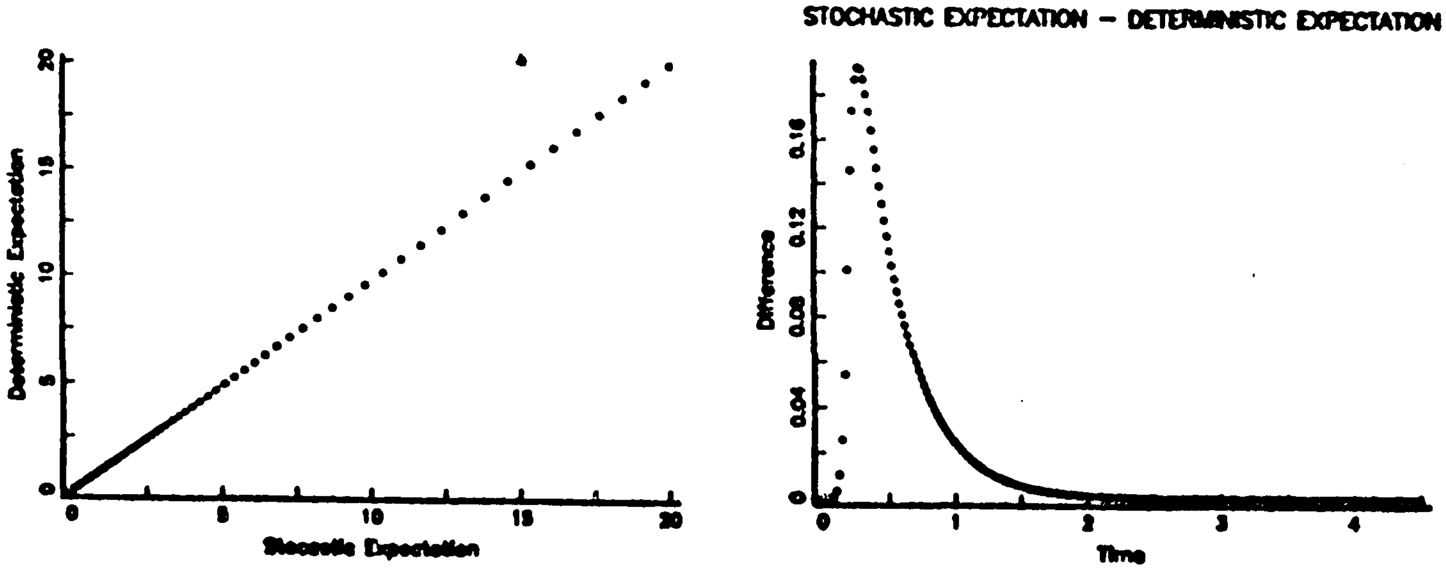


Figure 4.7. Bias of the Deterministic Mean: I Scenario

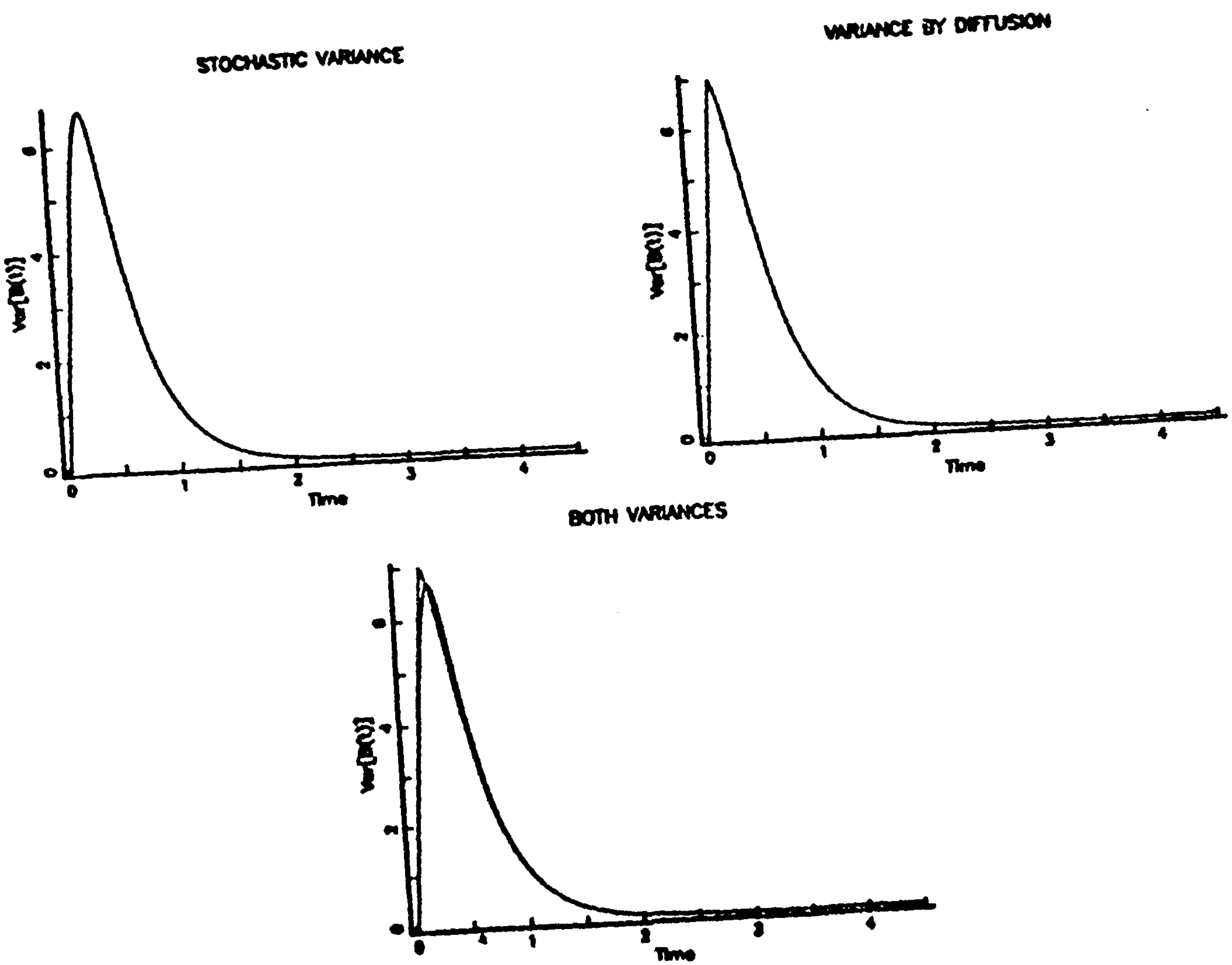


Figure 4.8. Stochastic Variance Versus Variance by Diffusion Approximation: I Scenario

stochastic variance versus the diffusion variance, and of the difference between both variances versus time, are shown in Figure (4.9). It is obvious from the figure that there exists some bias in the variance, and such bias tends to disappear as $t \to \infty$. In this case, the bias is defined to be the difference between the stochastic variance and the diffusion variance.

Bias = Stochastic Variance - Variance by Diffusion

Since there exists bias in both the mean and variance approximations, we expect to have some difference between the actual probabilities for the Markov process and the probabilities resulting from applying the normal approximation.

Figure (4.10) shows a comparison between the actual values for $P\{B(t) > 2 | B(0), D(0)\}$ as a result of applying the Markov-process model, and the normal approximation values without continuity correction using the diffusion approximation. Figure (4.11) shows the same comparison as Figure (4.10) but after applying the continuity correction factor to the diffusion approximation. We find that the continuity correction factor has improved the diffusion approximation results.

Figure (4.12) shows a comparison between the actual values of $P\{B(t) > 2 | B(0), D(0)\}$ and the large deviations approximation at large values of t. There is no continuity

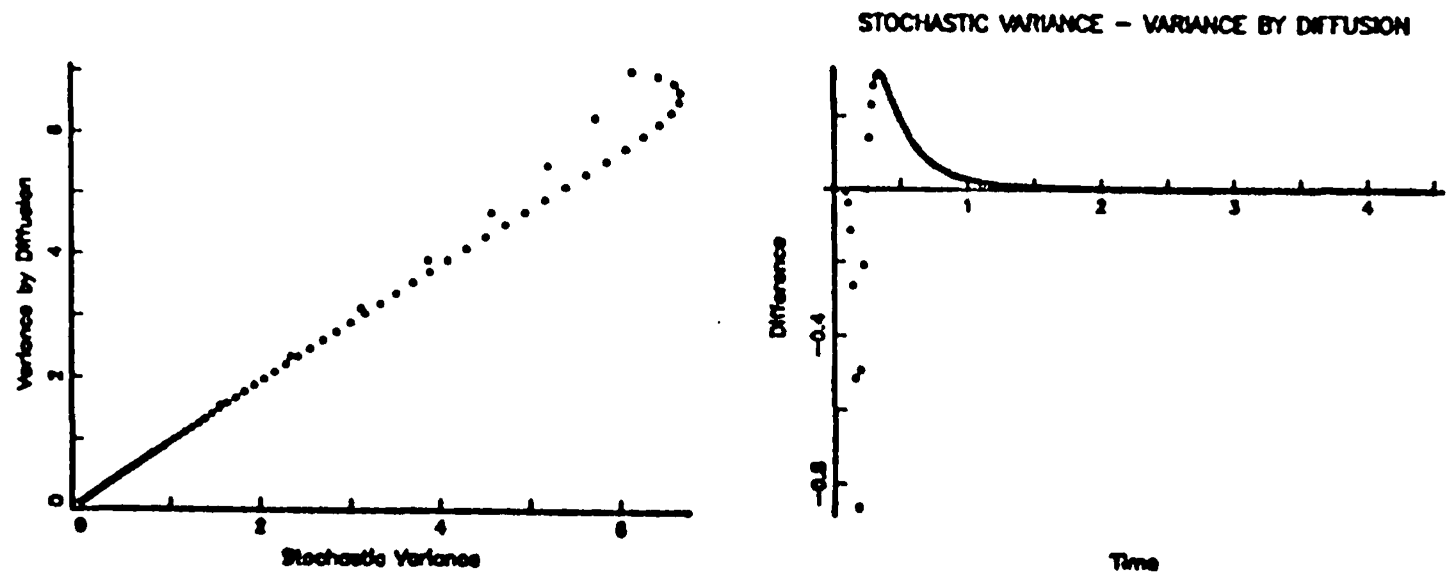


Figure 4.9. Bias of the Variance by Diffusion Approximation: I Scenario

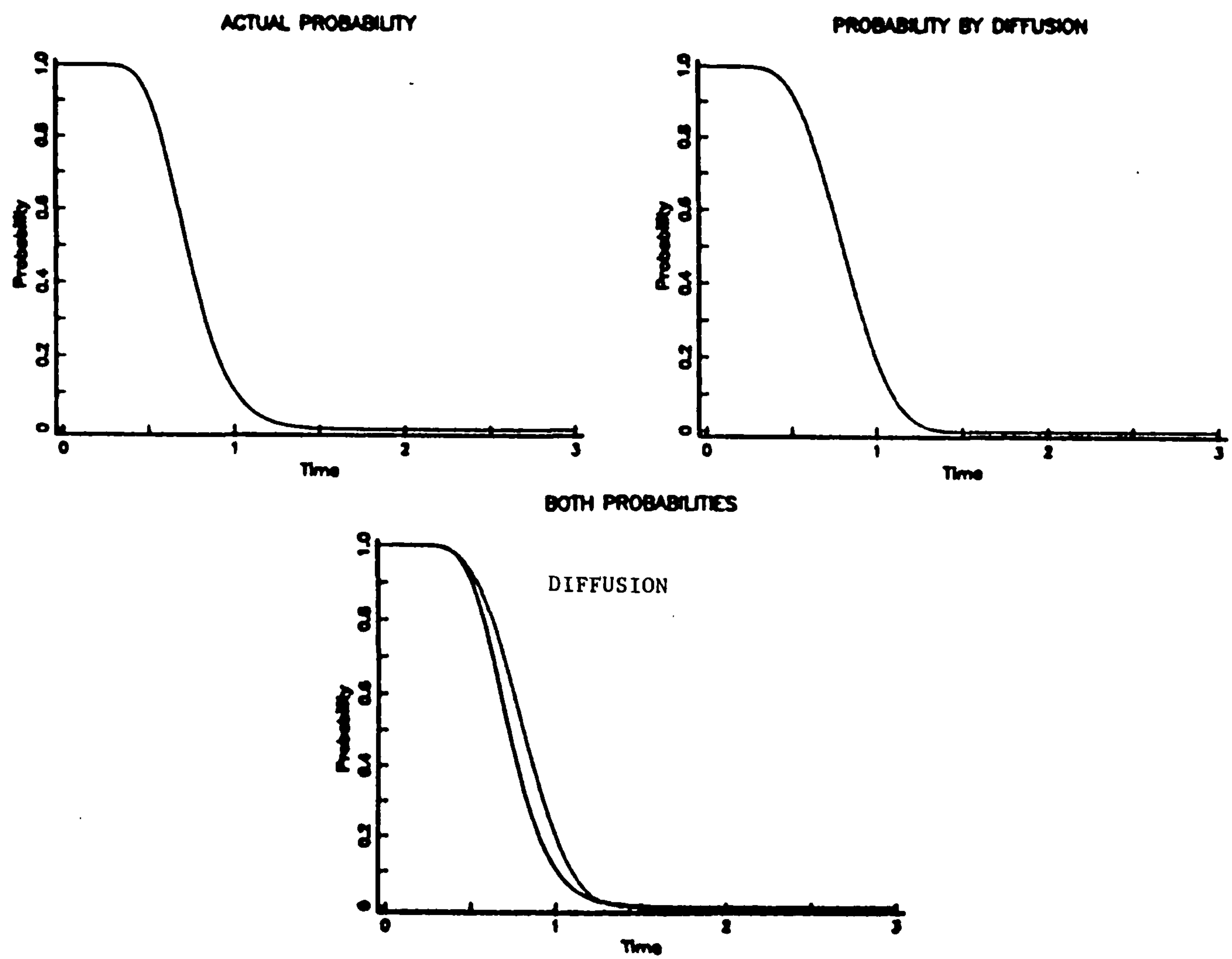


Figure 4.10. Comparison of Actual and Diffusion Probabilities without Continuity Correction: I Scenario

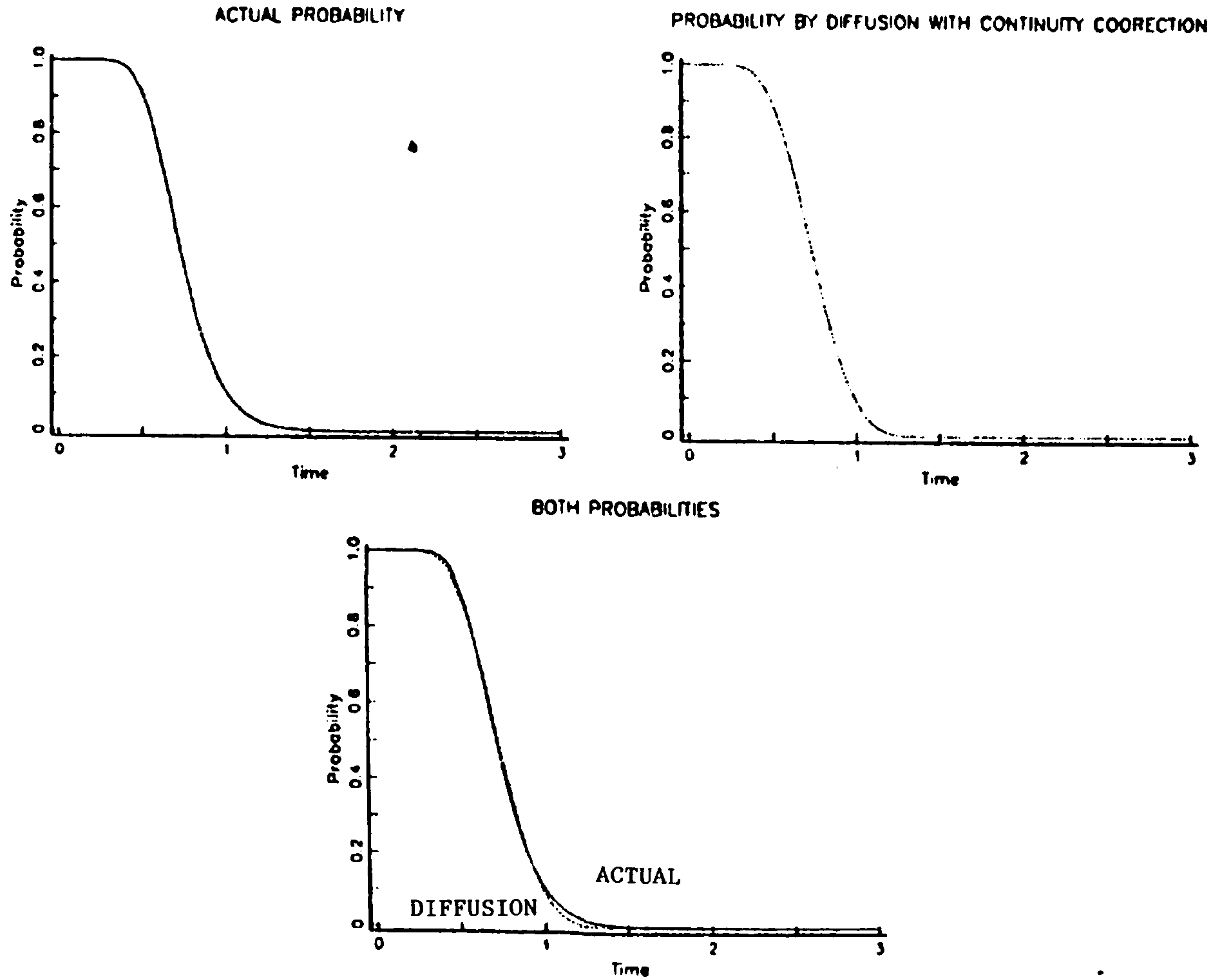


Figure 4.11. Comparison of Actual and Diffusion Probabilities with Continuity Correction Factor: I Scenario

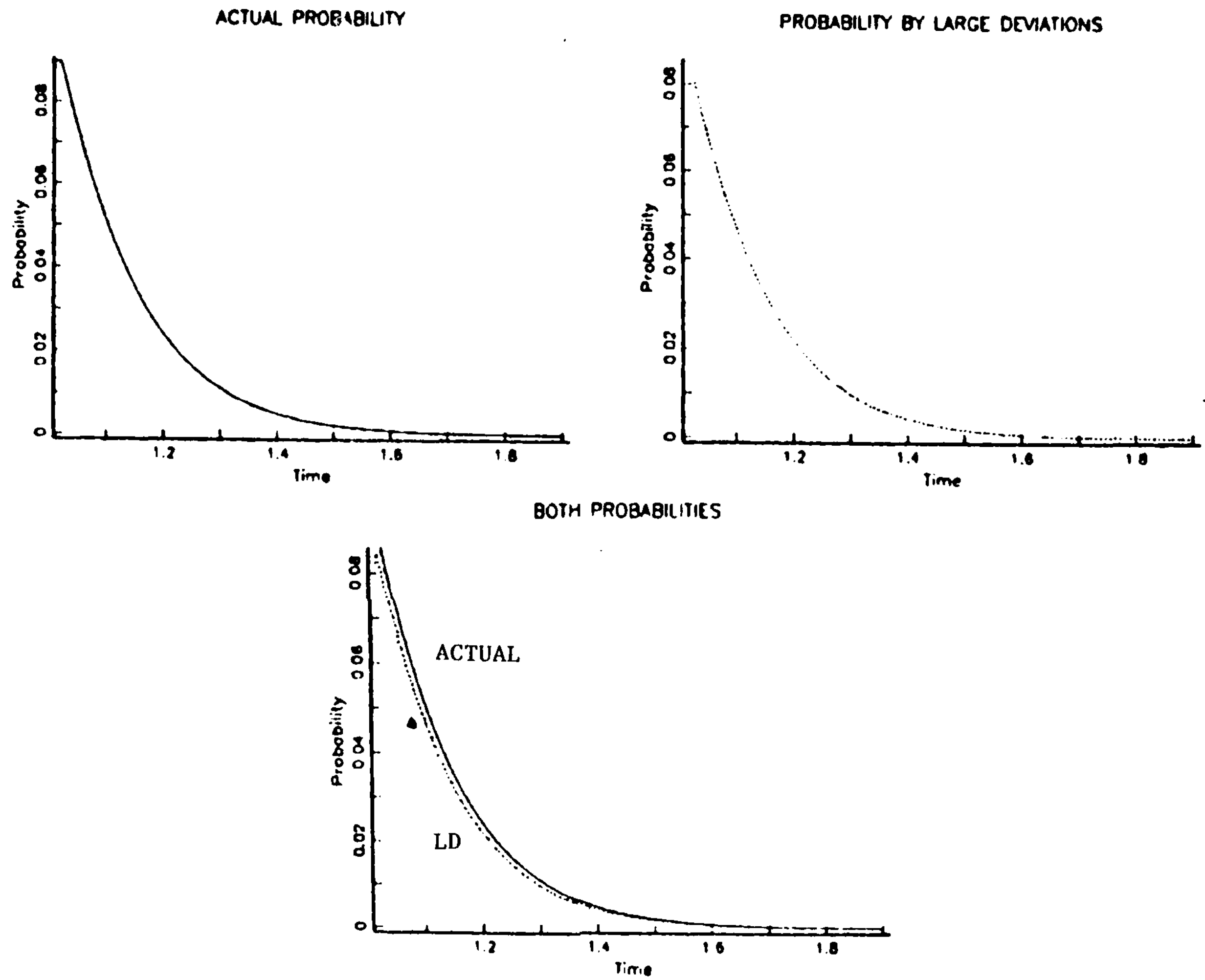


Figure 4.12. Comparison of Actual and Large Deviation Probabilities, for I Scenario

correction for the large deviations, since the large deviation equation has been derived for a sum of exponential random variables. Figure (4.13) shows a comparison of the actual probability and the diffusion approximation for large values of t. Comparing Figures (4.12) and (4.13), we find that, for large values of t, the large deviations better approximate the actual probability than the diffusion approximation. Figure (4.14) shows a plot of the diffusion approximation versus the actual probability for large values of t. We find that the resulting curve doesn't form a 45° straight line. Figure (4.14) also shows a plot of the relative error over time, where

$$\text{relative error} = \frac{|\text{Actual} - \text{Diffusion}|}{\text{Actual}}$$

We find that the error tends to increase steadily until it hits a 100% error, a 100% error means that the diffusion has resulted in $P\{B(t) > 2 | B(0), D(0)\} = 0$, while the actual probability is not yet equal to zero.

Figure (4.15) shows a plot of the large deviations approximation versus the actual probability. We find that the resulting plot forms approximately a 45° straight line, which suggests that the large deviations yield a good approximation to the actual probability. Figure (4.15) also shows a plot of the relative error over time, where the relative error, in this case, is defined as:

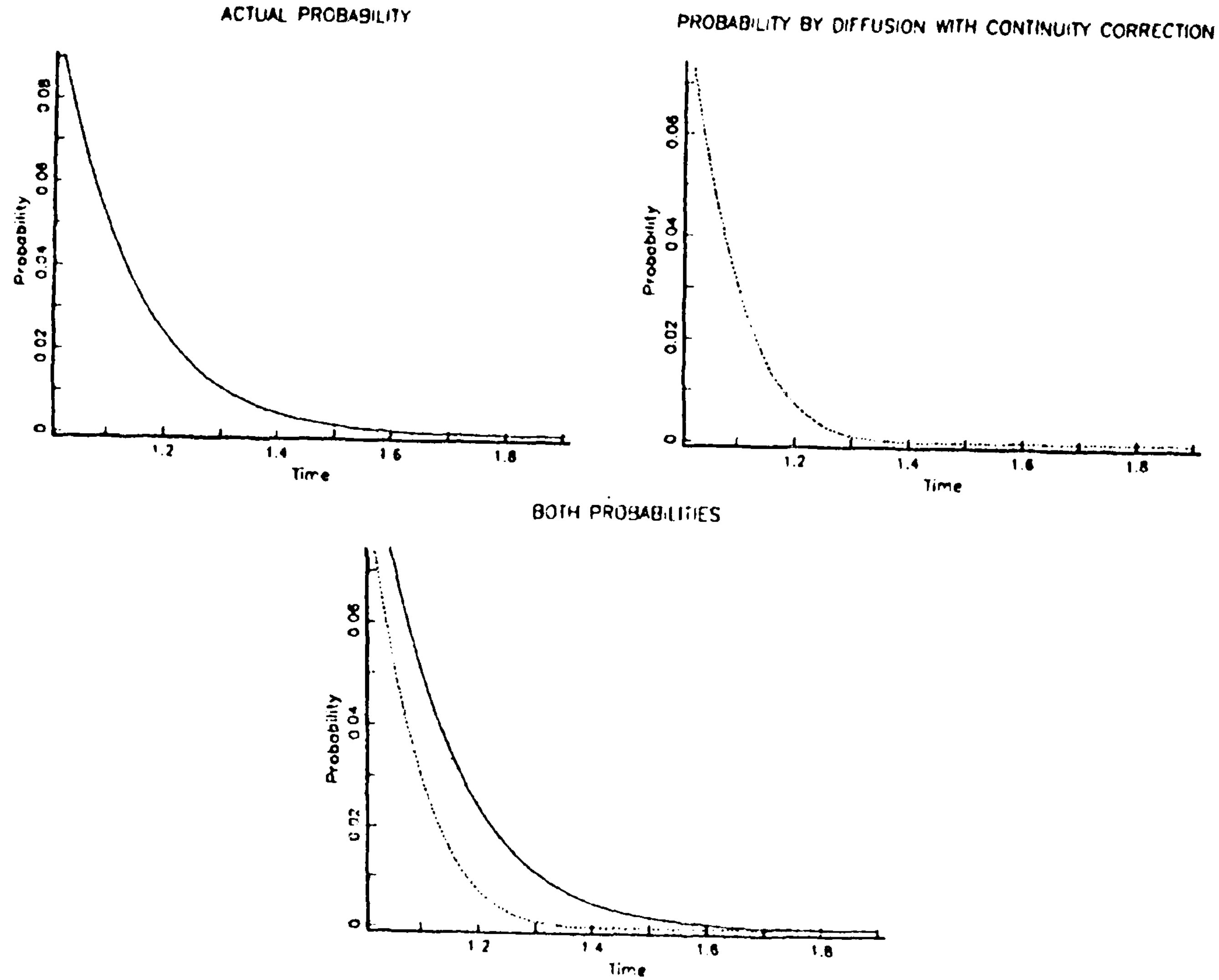


Figure 4.13. Comparison of Actual and Diffusion Approximation for Large Values of t

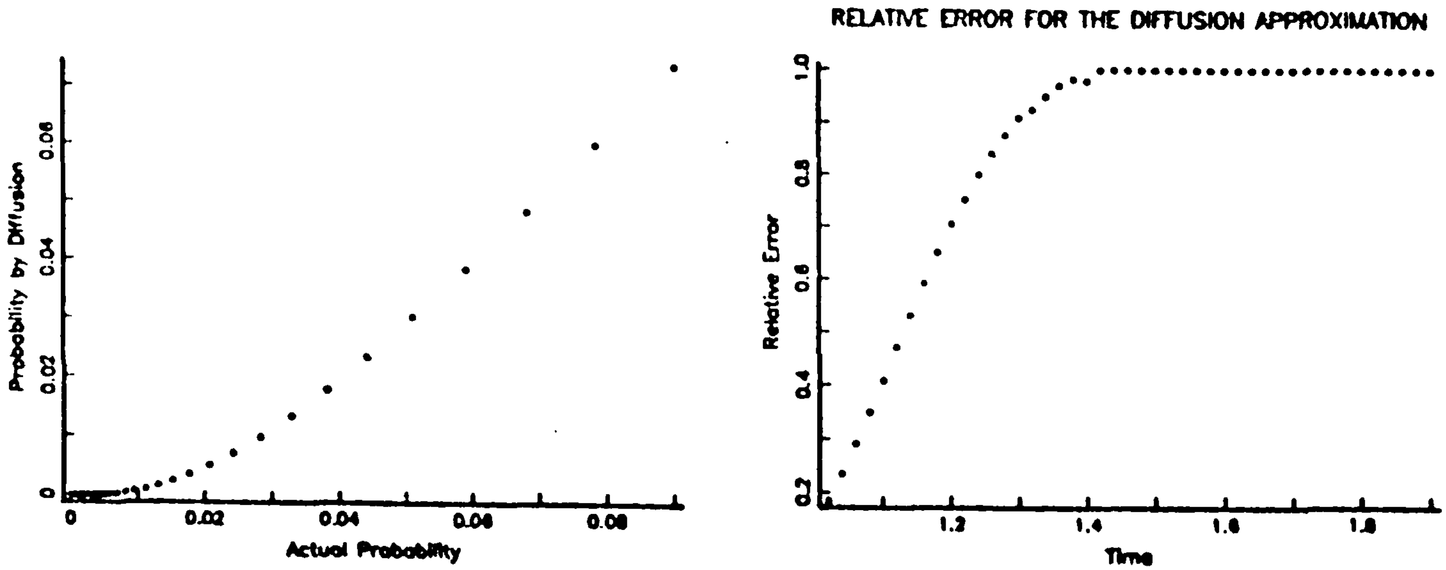


Figure 4.14. Error Analysis for the Diffusion Approximation

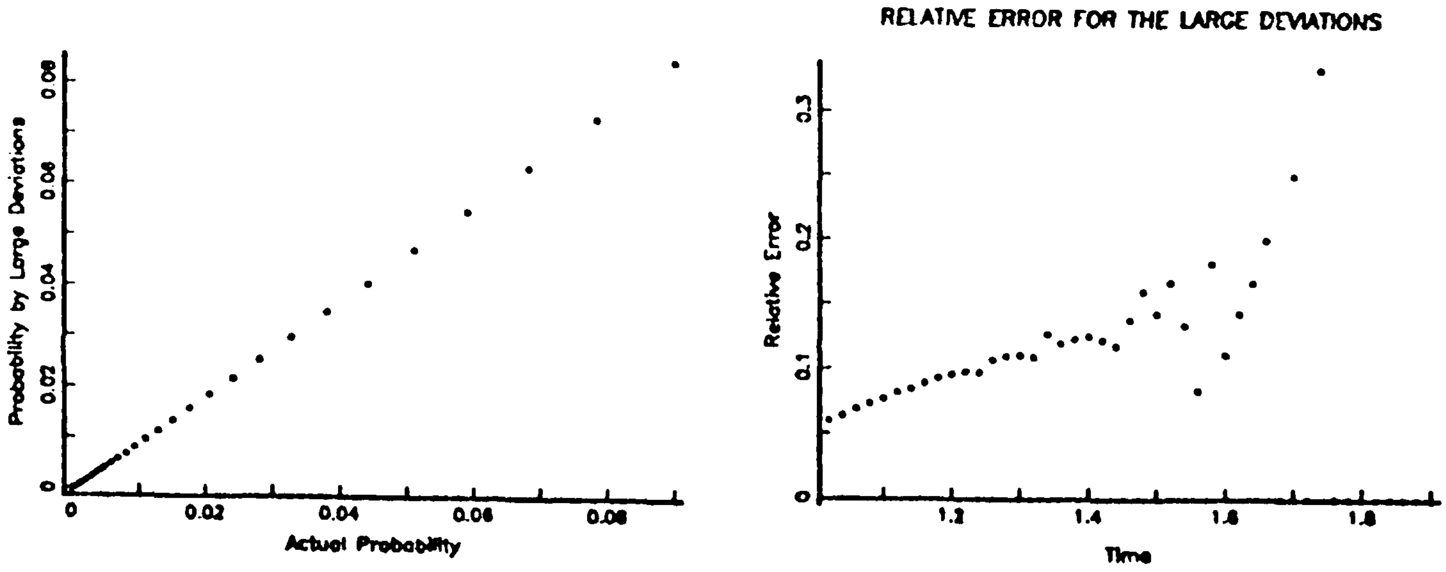


Figure 4.15. Error Analysis for the Large Deviations Approximation

$$\text{Relative Error} = \frac{|\text{Actual} - \text{Large Deviations}|}{\text{Actual}}$$

We find that the error tends to increase initially but drops to zero for very large values of t. This suggests that the large deviations tend to give a better approximation for large values of t (i.e., for values that are in the far tail).

V. COMBAT-LOGISTICS MODELS

A. INTRODUCTION

One of the major problems an air force faces is the determination of its budget allocation when a new weapon system is to be phased in. In this chapter, the term "weapon system" means that air force system which consists of the following:

- Aircraft.
- Spare parts.
- Repair facilities, including personnel.
- Early warning system.

Other components, such as the intelligence system, and the communication, command and control (C^3) system can be considered to be part of the general weapon system defined above. For example, as seen in the operational logistics models constructed in Chapter II, the effect of the intelligence system is considered under the total and partial surprise scenarios. The C^3 system effect is introduced very economically in the combat models under the BCD scenario, developed in Chapter IV, which is represented by the time spent to detect and engage an enemy bomber. The early warning system influences the duration of the combat. The sooner the attack is detected, the sooner the defenders can meet and engage it before it reaches the handover or bomb release line.

The outcome of combat depends upon the number of defenders ready to engage at the time of attack, D(0). For planning purposes, D(0) is a random variabe; its distribution has been determined in Chapters II and III by use of the logistics model.

The effectiveness of a weapon system, WS, must be evaluated by a model that determines the readiness of WS at the time of attack, and then computes the MOE for the squadron. Hence, we need a model that is capable of integrating a logistics model with a combat model. Such a model is referred to as a Combat-Logistics model.

In formulating combat-logistics models, the type of aircraft is assumed to be known to the analyst. In reality the analyst may wish to assess the effectiveness of different aircraft types. Mathematical optimization techniques may finally be applied to the combat-logistics model to define the optimal weapon system.

B. OBJECTIVES

The main objective of this chapter is to determine an optimal weapon system for an interceptor squadron that meets specified mission needs. The determination takes into account budget, combat, and logistics constraints.

To determine an optimal weapon system means first to identify the various feasible weapon systems that are available for the decision maker, considering "real-life" constraints, and then to select the weapon system that optimizes the

measure of effectiveness of the interceptor squadron. In reality it may be best to present the decision maker with a selection among a few systems (a short menu) along with some uncertainty assessments as to cost and performance.

C. SCOPE

An air-interceptor squadron is selected to represent the situation under study. The squadron is considered to consist entirely of the same type of interceptor (or fighter) aircraft. The decision maker has a budget of $K for procuring the weapon system, and an annual budget of $\$K_1$, to be allocated for maintenance facilities (repairmen or crews, and equipment).

This study concentrates on finding the optimal way of distributing (allocating) the initial budget and the annual budget among the different components of the weapon system. Here the term "optimal" refers generally to the minimization of the probability of enemy bomber's penetration. The probability of penetration has been analyzed in the previous chapter.

D. THE MODEL'S BASIC CONCEPT

1. General

The CL model assumes a small or middle-sized air force, represented by a squadron, defending a specific area of responsibility against a known enemy. However, the mathematical modelling is more general and can be applied to many different situations of varying force size and composition.

2. Specific Assumptions

The following specific assumptions can be made:

(i) The threat can be assessed far into the future. The number of enemy bombers attacking, together with their capabilities, can be predicted.

(ii) There is no attrition of aircraft and/or modules during non-combat periods.

(iii) No modules leave or enter the logistics system.

(iv) There is only one level of repair (i.e., no depot repair).

(v) Immediate replacement of flight-line level.

(vi) Full instantantous cannibalization.

(vii) There is no recruiting or attrition of repairmen (or crews) during the non-combat periods.

(viii) There is no support from any of the neighboring country's air force during the combat periods.

E. APPROACH

To achieve the above objectives, the combat-logistics process of the squadron has to be analyzed. The combat-logistics process represents the combat, non-combat activities of the squadron. The effect of the budget allocation supplies the restrictions for determining the optimal weapon system. When a weapon system is procured, it will operate under a peacetime environment (non-combat periods), where it should be managed and maintained so that it is found in maximum readiness when called upon for combat. The weapon system then performs its prime objective (conducts combat), so as to achieve certain combat objectives, as measured by a

suitable MOE. Therefore, when comparing weapon systems, we need to evaluate the weapon system under both peace and wartime (combat and non-combat) environments.

WS is first evaluated during non-combat periods by the operational logistics models discussed in Chapters II and III. The readiness (availability) of aircraft is then determined as an initial condition for the warning model. If a total surprise scenario is considered then the distribution of aircraft readiness, resulting from considering the limiting distribution of the operational logistics model, is provided as an initial condition for the combat model. If a partial surprise scenario is considered, then the limiting distribution of the operational logistics model is considered to be the initial condition for the warning model, constructed in Chapter II. The warning model provides the distribution of aircraft readiness at time of attack. This is used as an input to the combat model, where MOE's are then evaluated. A schematic for the above approach is shown in Figure (5.1).

The combat-logistics process can be formulated as a mathematical programming problem. The constraints for the mathematical program are divided into two categories.

1. Budget Constraints

The budget constraints are subdivided into the following two categories:

(a) Fixed budget constraint: This represents the total budget allocated for procuring WS.

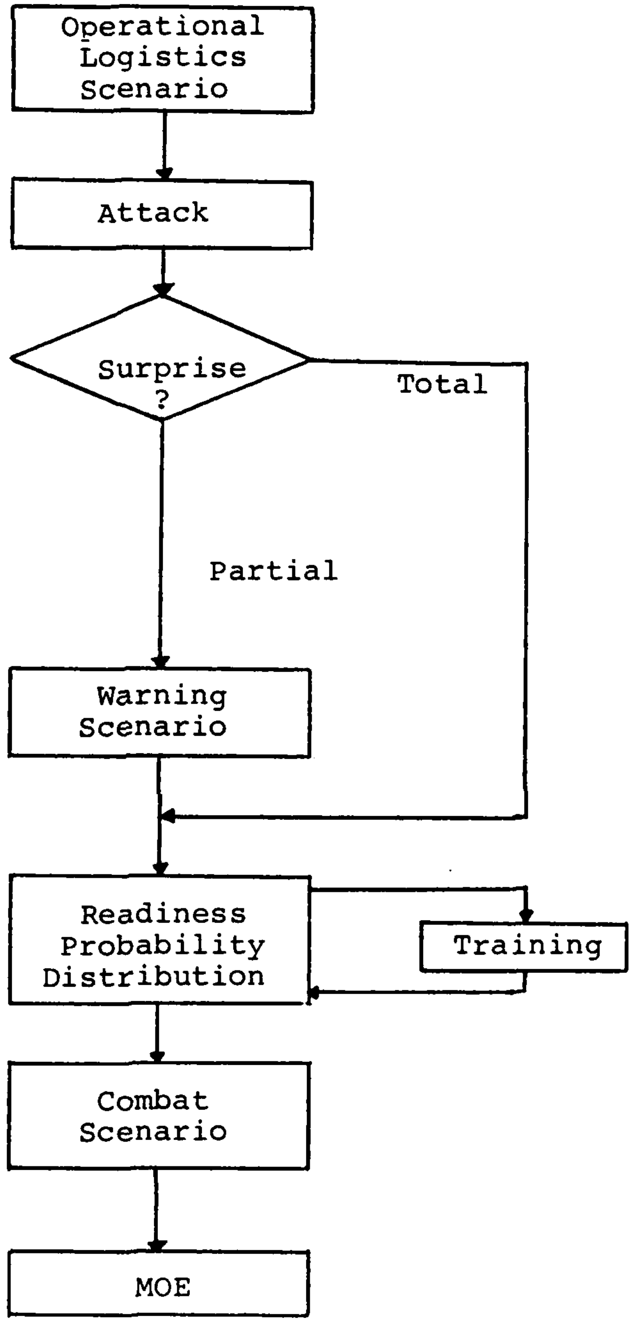


Figure 5.1. A Schematic for a Combat-Logistics Model

(b) Annual budget constraint: This represents the annual budget allocated for the squadron to operate and maintain WS.

2. Operational Constraints

These represent the necessary levels for the various components of WS, so that the squadron can conduct its mission during both combat and non-combat periods. Note that the budget constraints may be incompatible with the operational constraints (requirements).

F. PROBLEM FORMULATION

We now consider a simplified illustrative representation of the problem confronting a decision maker. Consider an aircraft squadron that is assigned a budget of \$K to procure WS. WS is to consist of:

1. A number, a, of aircraft.
2. M_1 and M_2 repairable modules of Type 1 and Type 2, respectively.
3. R_1 and R_2 repairmen to be assigned to Shop 1 and Shop 2, respectively.
4. An early warning system.

Suppose the squadron is allocated \$$K_1$ annually as a maintenance budget (or annual salaries for repairmen (plus the cost of necessary equipment).

The squadron is to defend the area against a prototypical threat of B(0) enemy bombers, attacking simultaneously a number of national and/or military targets. The squadron is assigned the responsibility of minimizing the probability

of the enemy bombers' penetration to or beyond their bomb release line (a hand-over line). Recall the following combat model. Each enemy bomber is assumed, when engaged by a defender, to take an exponential amount of time, with mean α_B^{-1}, to kill a defender (if unopposed). Each defender is assumed to take an exponential amount of time, with mean θ^{-1}, to detect and engage a free bomber, and an exponential amount of time, with mean α_D^{-1}, to kill a bomber (if unopposed) once engagement starts; the combat is thus modelled as a competing risk problem. It is assumed that only one free defender can engage a free bomber, and that the engagement will be fight-to-the-finish.

Let λ denote the overall Markovian failure rate of an individual aircraft, and let p_1 denote the conditional probability that a failure requires just Type 1 (e.g., engines) repair, p_2 the conditional probability that the failure is of Type 2 (e.g., avionics) and p_{12} the conditional probability that both Type-1 and Type-2 failures occur. Thus, $p_1 + p_2 + p_{12} = 1$, and the sequence of successive failure types is one of independently and identically distributed random variables. Finally, it is assumed that repair is Markovian (or exponential), where only one repairman can work on a failed module, and μ_i denotes the rate at which an individual repair of Type i, i = 1,2, is completed at Shop i. Let R_i be the number of repairmen assigned to Shop i, i = 1,2.

Recall that:

$X_i(t)$, $i = 1,2$, denotes the number of modules of Type i, $i = 1,2$, that are in or require repair at time t.

$B(t)$ denotes the number of free bombers at time t.

$C(t)$ denotes the number of bombers (hence defenders) in engagement at time t.

$D(t)$ denotes the number of free defenders at time t.

As shown in previous chapters, the stochastic process $\{X_1(t), X_2(t);\ t \geq 0\}$ is a bivariate birth and death process; and the stochastic process $\{B(t), C(t), D(t);\ t \geq 0\}$ is a trivariate pure death process, defined by (2.16) and (4.45), respectively.

It was shown in Chapter IV that the process $\{B(t), C(t), D(t);\ t \geq 0\}$ can be approximated by the trivariate diffusion process $\{\tilde{B}(t), \tilde{C}(t), \tilde{D}(t);\ t \geq 0\}$; where $\tilde{B}(t)$, $\tilde{C}(t)$, and $\tilde{D}(t)$ is the continuous approximation for $B(t)$, $C(t)$, and $D(t)$, respectively.

G. MODEL'S STRUCTURE

1. The Objective Function

The objective of the squadron is to minimize the probability of penetration of enemy bombers. The probability of more than z free bombers reaching their bomb release line is given by $P\{B(t) > z | B(0)\}$; this can be obtained by blending the combat model with the logistics model. From the law of total probability, we obtain:

$$P\{B(t) > z | B(0)\} = \sum_{j=0}^{a} P\{B(t) > z, D(0) = j | B(0)\}$$

It follows that,

$$P\{B(t) > z | B(0)\} = \sum_{j=0}^{a} P\{B(t) > z | B(0), D(0) = j\} P\{D(0) = j\} \qquad (5.1)$$

Let $\mu(B(0),D(0),t)$ and $\sigma^2(B(0),D(0),t)$ be the conditional mean and variance of the marginal process $\{B(t);\ t \geq 0\}$, given the initial force levels. From Results (4.2), and (4.3), we find that:

$$\tilde{B}(t) \sim N(\mu(B(0),D(0),t); \sigma^2(B(0),D(0),t))$$

It follows that $P\{B(t) > z | B(0) = B\}$ can be conveniently calculated as follows:

$$P\{B(t) > z | B(0) = B\} \simeq \sum_{j=0}^{a} P\{\tilde{B}(t) > z | B(0) = B, D(0) = j\} P\{D(0) = j\}$$

$$= \sum_{j=0}^{a} \{1 - \Phi\{\frac{z - \mu(B,j,t)}{\sigma(B,j,t)}\}\} P\{D(0) = j\} \qquad (5.2)$$

Thus,

$$P\{B(t) > z | B(0) = B\} \simeq 1 - \sum_{j=0}^{a} \Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\} P\{D(0) = j\}$$

where:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

Hence, the objective function can be approximated by,

$$\min_{WS} \left[1 - \sum_{j=0}^{a} \Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\} P\{D(0) = j\} \right],$$

or equivalently by:

$$\max_{WS} \left[\sum_{j=0}^{a} \Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\} P\{D(0) = j\} \right] \qquad (5.3)$$

where $\min_{WS}[\quad]$ refers to the act of minimizing (maximizing) the quantity [ ], subject to budget constraints.

To evaluate the objective function, we need, for every j, to:

(a) evaluate the term $\Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\}$ using one of the diffusion combat models constructed in Chapter IV. Alternative combat representation can also be utilized as required, but the diffusion model renders computer calculations economical.

(b) evaluate the term $P\{D(0) = j\}$ using one of the operational logistics models constructed in Chapters

II and III for a range of logistics assets (spare parts, repair facilities) that is within budget constraints.

(c) select the optimal combination from the results of (b).

2. Budget Constraints

a. Fixed Budget Constraint

A major constraint when a weapon system is to be procured is the amount of available (or planned) budget. The budget is thought of, in this study, as a fixed budget that is available to purchase the hardware components of the weapon system, which are:

- aircraft,
- modules of Type i, $i = 1,2,\ldots,I$
- early warning system

where:

I = number of different types of modules

Hence, the budget must be equal to or exceed the summation of the products of the above components of WS times their individual cost coefficients, assuming that linear costs are appropriate; otherwise a suitable non-linear cost is required. Thus:

$$c_0 a + \sum_{i=1}^{I} c_i M_i + \sum_{n=1}^{N} d_n E_n \leq K \qquad (5.4)$$

where:

$K(\$)$: Total amount of budget available

M_i : Number of Modules of Type i

E_n : Early warning system, type n

c_o, c_i, d_n: The cost coefficients of aircraft, module type i, and n^{th} early warning system respectively

N : Number of different types of early warning systems available to purchase.

b. Annual Budget Constraint

The annual budget considered, for this study, is the annual maintenance budget. This is represented by considering the annual running cost of the maintenance facilities (e.g., annual salaries of repairmen).

Therefore, the planned annual budget must be equal to or exceed the summation of the products of the number of repairmen (or crews) assigned to Shop i times their individual annual salary (or cost). Thus we obtain:

$$\sum_{i=1}^{I} r_i R_i \leq K_1$$

where:

K_1 = Total annual budget planned for maintaining WS

R_i = Total number of Repairmen to be assigned to Shop i

r_i = The annual salary for an individual repairman assigned to Shop i.

3. <u>Operational Constraints</u>

In order for the interceptor squadron to conduct training programs and reconnaissance missions during non-combat periods, and to be able to cause some damage to enemy bombers during combat periods, the number of aircraft, a, must have a positive lower bound. The lower bound of a, ℓ_a, should be at least 2 aircraft so that some training missions can be conducted. However, the solution strategy for our models allows different values of ℓ_a which can be determined by the decision maker. Therefore, we obtain the following condition:

$$a \geq \ell_a \tag{5.6}$$

Similarly, lower bounds for M_i, $i = 1,2,\ldots,I$, can be derived. In order for the event of having all of the aircraft assigned to the squadron operational, to occur, we must have the following condition:

$$M_i \geq a \qquad i = 1,2,\ldots,I \tag{5.7}$$

Since the modules are assumed to be repairable, then every shop should be assigned at least one repairman (or crew). Of course our model implies that the probability of having at least one idle repairman at Shop i is one if $R_i > M_i$. Hence we obtain the obvious constraint:

$$1 \le R_i \le M_i \,, \qquad i = 1,2,\ldots,I \tag{5.8}$$

In a more detailed model several crews or individuals might work simultaneously.

Modules that are operational, and not installed on any aircraft, are stored in the supply depot. This results in a capacity constraint; for planning purposes, this can be represented by the decision maker specifying a certain coefficient, ρ_i, for determining the ratio of modules of Type i to aircraft, where:

$$M_i \le \rho_i a \quad \text{then} \quad \rho_i \in [1,\infty)$$

Considering (5.7), it follows that,

$$a \le M_i \le \rho_i a \qquad i = 1,\ldots,I \tag{5.9}$$

It is assumed that only one early warning system is to be purchased for the air defense system. This can be represented by the following constraint:

$$\sum_{n=1}^{N} E_n = 1 \,, \quad \text{and } E_n \text{ is binary, } n = 1,\ldots,N \tag{5.10}$$

Finally the WS, WS*, that maximizes the objective function and satisfies the constraints is the solution of the following non-linear-integer-mathematical-programming problem.

$$\max_{WS} \sum_{j=0}^{a} \Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\}P\{(DO) = j\}$$

s.t.

$$c_0 a + \sum_{i=1}^{I} c_i M_i + \sum_{n=1}^{N} d_n E_n \leq K$$

$$\sum_{i=1}^{I} r_i R_i \leq K_1$$

$$\sum_{n=1}^{N} E_n = 1$$

$$a \geq \ell_a$$

$$a \leq M_i \leq \rho_i a \qquad i = 1,\ldots,I$$

$$1 \leq R_i \leq M_i \qquad i = 1,\ldots,I$$

$$E_n \text{ binary} \qquad n = 1,\ldots,N$$

a, M_i, R_i, $i = 1,\ldots,I$, are positive integers

In general models that integrate logistics with combat will be referred to as combat-logistics (CL) models.

H. COMBAT LOGISTICS MODELS: SIMPLE ILLUSTRATIONS

Suppose there are two early warning systems, E_1 and E_2, available for purchasing. Each system can allow a different combat duration. The term "combat duration," referred to in this chapter, is the length of time required for the bombers to reach their bomb release line, measured from the beginning of the engagement.

Let t_1 be the combat period achieved by deploying E_1, and t_2 be the combat period achieved by deploying E_2; t_1 and t_2 are assumed to be deterministic.

Special cases of the above combat-logistics model are the Single-Module Combat-Logistics (1MCL) and the Two-Module Combat Logistics (2MCL) models.

1. Single-Module Combat-Logistics Model (1MCL)

Under the single-module logistics assumptions, stated in Chapter II, the corresponding combat model becomes:

$$\text{1MCL:} \quad \max_{WS} \sum_{j=0}^{a} \Phi\left\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\right\} P\{D(0) = j\}$$

s.t.

$$c_0 a + cM + d_1 E_1 + d_2 E_2 \leq K$$

$$E_1 + E_2 = 1$$

$$a \geq \ell_a$$

$$a \leq M \leq \rho a$$

E_1, E_2 binary; a, M positive integers.

Note in the present 1MCL formulation we may drop the annual budget constraint for the repairmen. This is because it is assumed that there is only one shop. Therefore, the number of repairmen, R, to be assigned to the repairshop will be:

$$R = \left\lfloor \frac{K_1}{r} \right\rfloor$$

where r = annual salary for an individual repairman, since the above salaries are the only component of annual cost in the present model.

The maximization is thus over the specification of WS, where we abbreviate the latter as

$$WS = [a, M, E_1]$$

i.e., WS is represented by a vector whose components are the number of aircraft, a, the number of spare modules, M, and the particular warning system selected, E_i. If $E_1 = 1$ then the first early warning system is selected, otherwise $E_1 = 0$ which indicates that the second early warning system is selected.

Availability of aircraft is a nondecreasing function of the number of modules purchased for a given number of aircraft. That is, if the squadron has a number, a, of aircraft then procuring more modules will tend not to decrease

the availability of aircraft. The above, and constraints of 1MCL, justify the following, for a given value of a:

$$\text{let} \quad M = \min\{\rho a, \left\lfloor \frac{K-d_1E_1-d_2E_2-c_0a}{c_1} \right\rfloor\}$$

1MCL can be solved iteratively over all feasible values of a, E_1, and E_2; while choosing M to be as given above.

2. Two-Module Combat-Logistics Model (2MCL)

Under the two-module lgistics assumptions, stated in Chapter II, the corresponding combat model becomes:

$$\text{2MCL:} \qquad \max_{WS} \sum_{j=0}^{a} \Phi\{\frac{z-\mu(B,j,t)}{\sigma(B,j,t)}\}P\{D(0) = j\}$$

s.t.

$$c_0a + c_1M_1 + c_2M_2 + d_1E_1 + d_2E_2 \le K$$

$$r_1R_1 + r_2R_2 \le K_1$$

$$E_1 + E_2 = 1$$

$$a \ge \ell_a$$

$$a \le M_1 \le \rho_1 a$$

$$a \le M_2 \le \rho_2 a$$

$$1 \le R_1 \le M_1$$

$$1 \le R_2 \le M_2$$

E_1, E_2 binary; a, M_1, M_2, R_1, R_2 are positive integers.

In this case, WS is specified by the vector

$$WS = [a, M_1, M_2, R_1, R_2, E_1]$$

In the k-module, s-shop case the WS is specified by an analogous vector.

As in the case of the 1MCL, for a given a and M_1, it is assumed that the availability of aircraft is a non-decreasing function of M_2. Also, for a given a, M_1, M_2, and R_1, we assume that the availability of aircraft is a non-decreasing function of R_2. Therefore for a given value of a, M_1 and R_1, let

$$M_2 = \min\{\rho_2 a, \left\lfloor \frac{K-d_1E_1-d_2E_2-c_0a-d_1M_1}{c_2} \right\rfloor\},$$

and let

$$R_2 = \min\{M_2, \left\lfloor \frac{K_1-r_1R_1}{c_2} \right\rfloor\}$$

2MCL can now be solved iteratively over all feasible values of a, M_1, E_1, E_2 and R_1, while choosing M_2 and R_2 as given above.

I. SPECIFIC ILLUSTRATIONS

1. General

This section presents illustrations of some scenarios that can be modelled by 1MCL, and 2MCL. The scenarios

considered in this section are the BCD, and the I combat models, under the total surprise situation. We first present two illustrations of the 1MCL model. One uses the I model for the combat part of the model, and the other uses the BCD model. Then, we present an illustration of the 2MCL model that uses the I combat model.

The diffusion models constructed in Chapter IV are used to approximate the combat process under both the I and BCD models. The single-module logistics model, and the two-module logistics model, constructed in Chapter II are used to represent the logistics system required for WS.

2. Model Parameters: 1MCL

The following parameters are used for model illustrations:

(i) Threat. The threat considered is $B(0) = 12$ bombers attacking simultaneously. We have not taken into account the possible uncertainty in identifying the threat magnitude.

(ii) MOE. The decision maker is interested in minimizing the probability of more than 3 bombers penetrating, or reaching their bomb-release line.

(iii) Logistics. Each aircraft fails at a rate of $\lambda = 0.5$ aircraft per day.
- There are $R = 4$ repair crews assigned to the squadron.
- Each failed module is repaired at rate $\mu = 1.0$ modules per day, where only one repair crew can work on a failed module.
- There is enough space to store modules when they are up and not installed on aircraft; i.e., ρ is assumed to be ∞.

(iv) Combat. Each defender takes an exponential amount of time, with rate of $\alpha_D = 3$ (bombers per hour), to kill a bomber, where only one defender can engage a free bomber.

- For the I model the defenders are invulnerable, i.e., $\alpha_B = 0$.
- For the BCD model, C^3 takes an exponential amount of time, with rate $\theta = 0.6$ to detect and engage a free bomber. Each bomber, if engaged, takes an exponential amount of time with rate $\alpha_B = 0.5$ (defenders per hour), to kill a defender.
- If early warning system E_1 is deployed then the combat period is $t_1 = 1.0$ hour. If E_2 is deployed then the combat period is $t_2 = 0.8$ hour.
- The lower bound for the number of aircraft, ℓ_a, is 2.

(v) <u>Cost</u>.
- The squadron has been allocated a budget of $K = \$115$ million for purchasing the WS.
- Each aircraft costs $c_0 = \$10$ million.
- Each module costs $c = \$2$ million.
- The E_1 costs $d_1 = \$15$ million, while E_2 costs $d_2 = \$5$ million.

Using the above parameters, the 1MCL becomes:

$$\max_{WS} \sum_{j=0}^{a} \Phi\left\{\frac{z-\mu(12,j,t)}{\sigma(12,j,t)}\right\} P\{D(0) = j\}$$

s.t.

$$10a + 2M + 15E_1 + 5E_2 \leq 115$$

$$E_1 + E_2 = 1$$

$$a - M \geq 0$$

$$a \geq 2$$

$$E_i \text{ binary} \quad i = 1,2$$

$$a, M \text{ integer.}$$

This model was solved by evaluating all feasible WS.

Let

$$t = t_1 E_1 + t_2(1 - E_1)$$

Then, if E_1 is deployed then $E_1 = 1$ and $t = t_1$, otherwise E_2 is deployed and $t = t_2$ since $E_1 = 0$. For every j, $\mu(12,j,t)$ and $\sigma^2(12,j,t)$ are determined using Result (4.3) for the I model and Result (4.2) for the BCD model; where $\mu(\ )$ and $\sigma^2(\ )$ are the mean function and the variance function of the $\{B(t);\ t \geq 0\}$ process obtained by the diffusion approximations.

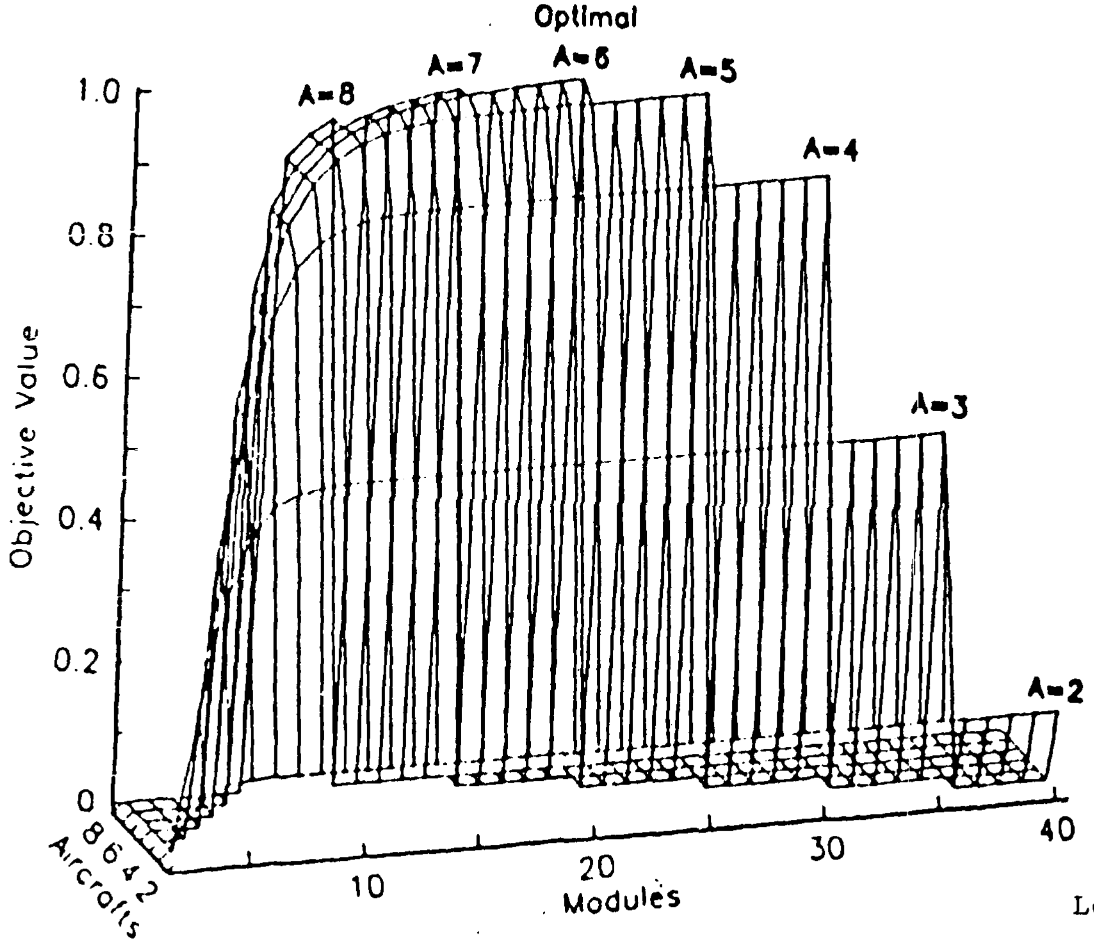
3. <u>1MCL Using I Model</u>

Under the I model assumptions, the diffusion approximation for B(t) is given by Result (4.3).

For every combination of a and M, $P\{D(0) = j\}$ was computed using equation (2.8).

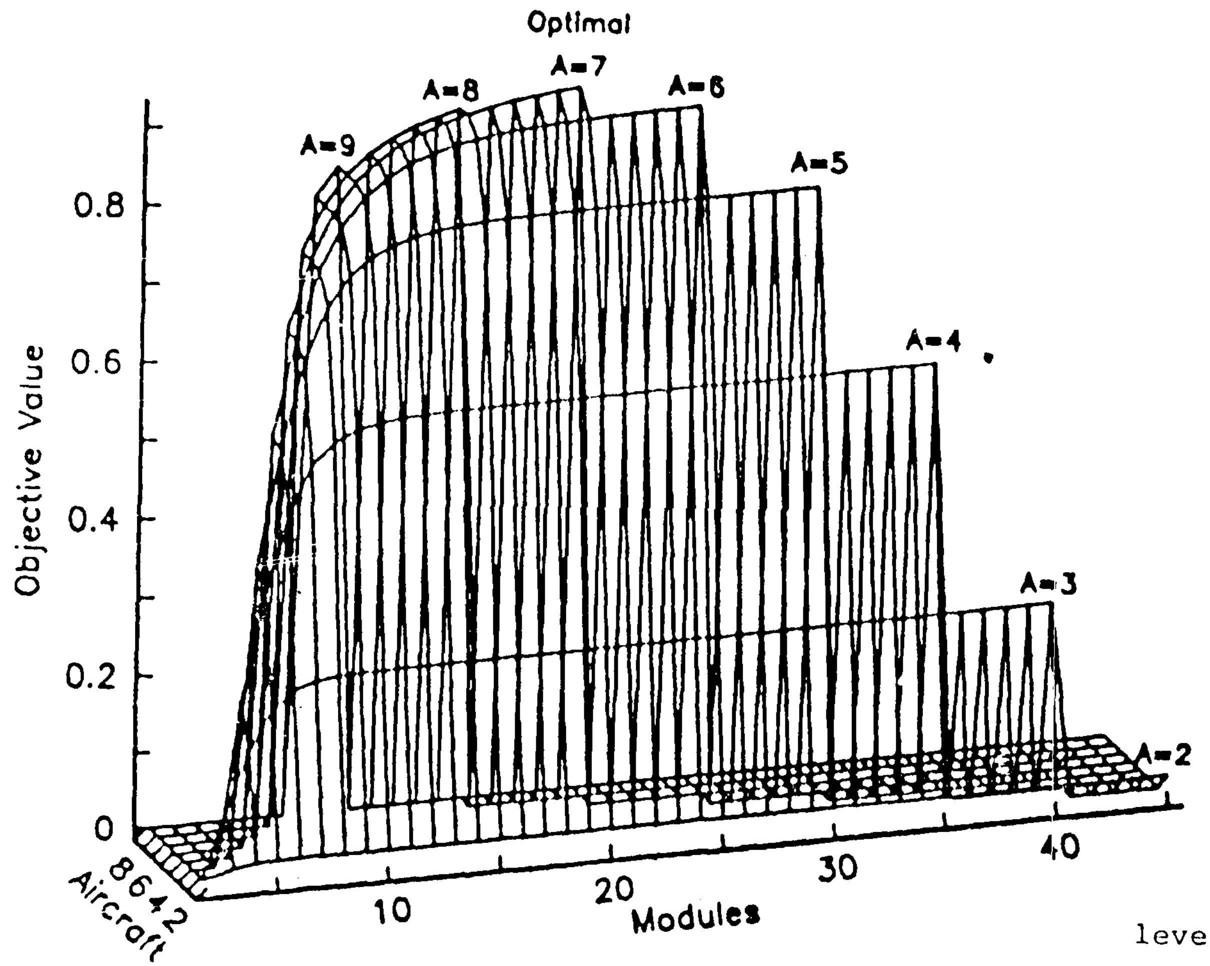
Figure (5.1) shows a surface plot of the objective function when $E_1 = 1$. The figure indicates that if E_1 is selected as the early warning system to be deployed, then the optimal WS = [6, 20, 1], i.e., to buy 6 aircraft, and 20 modules, this yields an objective value of 0.9903; i.e., a probability of penetration = 1 - 0.9903 = 0.0097.

Figure (5.2) shows a surface plot of the objective function when $E_2 = 1$, which indicates that the optimal solution will be to buy 7 aircraft and 20 modules, and obtain an objective value of .9211.



Level "A" = Number of aircraft

Figure 5.1. Surface Plot of the Objective Function for I Scenario with E_1 Deployed, Using SMCL(1)



level "A" = number of aircraft

Figure 5.2. Surface Plot of the Objective Function for I Scenario with E_2 Deployed: Using SMCL(1)

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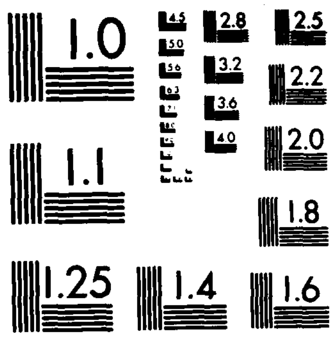

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

From the above, it is concluded that the optimal weapon system, WS*, is:

$$WS^* = [6, 20, 1]$$

i.e., buy 6 aircraft, 20 modules, and deploy E_1. WS* under the single-module logistics and the I-combat assumptions results in a probability of penetration of 0.0097.

a. Comparison with Exact Computations

If a = 6 aircraft and M = 20 modules, then we get the following readiness distribution, using (4.8):

j =	0	1	2	3	4	5	6
$P\{D(0) = j\}$ =	0	0.0002	0.0006	0.0017	0.0034	0.0054	0.9887

If we calculate $P\{B(1) \geq 3 | B(0) = 12, D(0) = j\}$, $j = 0,1,\ldots,6$, using the Markovian combat model, i.e., by solving the forward Chapman-Kolmogorov equations (4.58) numerically, we obtain the following:

j =	0	1	2	3	4	5	6
$P\{B(1) \leq 3 \| B(0) = 12, D(0) = j$ =	0	0.0004	0.1527	0.5444	0.7989	0.9590	0.9892

Therefore:

$$\sum_{j=0}^{6} P\{B(1) \geq 3 | B(0) = 12, D(0) = j\} P\{D(0) = j\} = 0.9870$$

It follows that by solving the Chapman-Kolmogorov equations numerically for WS* proposed by 1MCL, we obtain a probability of penetration of 0.0120. This yields a relative error of:

$$\left|\frac{.0097 - .0120}{.012}\right| = 0.19 ,$$

i.e., a relative error of about 20% is obtained. This indicates that the diffusion is <u>overly optimistic</u>. But, the order of magnitude seems adequate, even though the numbers are small.

4. <u>1MCL Using BCD Model</u>

When the BCD combat model was used with the parameters given earlier (i.e., $\alpha_D = 3$, $\alpha_B = .5$, $\theta = .6$), 1MCL gave: WS* = [7 15 1]; with an objective value of 0.5288, i.e., probability of penetration = 0.4712. If early warning system E_2, which has shorter detection range than E_1, is procured the corresponding optimal WS = [7 20 0], with an objective value of 0.4686. This shows that, for the current example, procuring E_1 results in a better combat effectiveness than procuring more modules.

Procurement of WS* = [7 15 1] yields a relatively high probability of penetration when compared with the I model case. This is due to introducing the combat parameters α_B and θ. Table (5.1) shows the probability of penetration for decreasing values of α_B, and increasing values of θ.

TABLE 5.1

BEHAVIOR OF PROBABILITY OF PENETRATION PROVIDED BY 1MCL AS θ INCREASES, AND $\alpha_B \rightarrow 0$

θ	α_B	Probability of Penetration
.6	.5	.47
.6	.2	.41
.6	.1	.38
.6	.05	.36
.6	.005	.35
.65	.005	.32
.8	.005	.24
.85	.005	.20
.95	.00005	.15
.95	0	.15
.99	0	.13
2	0	.03

We *find the probability* of penetration decreases as α_B decreases and θ increases.

It is evident from Table (5.1) that the MOE improves as θ increases and/or α_B decreases. In particular the MOE tends to approach the value provided by the 1MCL using the I model (i.e., .01).

a. Comparison with Simulation Results

From equation (4.8), we obtain the following readiness distribution for a squadron of 7 aircraft that has 15 modules, and 4 repair crews where the single-module logistics are assumed:

j	0	1	2	3	4	5	6	7
$P\{D(0) = j\}$ =	0.0002	0.0014	0.0056	0.0149	0.0298	0.0478	0.0637	0.8366

Since E_1 is deployed the combat duration is 1 hour. Therefore the MOE has to be evaluated at $t = 1$. Simulating the Markovian combat process $\{B(t), C(t), D(t); t \geq 0\}$ for 10,000 replications; where $\alpha_B = 0.5$ and $\theta = 0.6$ and collecting statistics at $t = 1$, we obtain the following values for $P\{B(1) \leq 3 | B(0) = 12, D(0) = j\}$, $j = 0,1,\ldots,7$:

j =	0	1	2	3	4	5	6	7
$P\ B(1) \leq 3 \| B(0) = 12, D(0) = j\}$ =	0.0	0.0	0.0	0.01	0.09	0.32	0.59	0.80

The simulation-of-combat calculation gives

$$\sum_{j=0}^{7} P\{B(t) \leq 3 | B(0) = 12, D(0) = j\} P\{D(0) = j\} = .72$$

i.e., probability of penetration = .28.

Recall that 1MCL gave a probability of penetration value of 0.47 which yields a relative error of about 67%. This indicates (as anticipated in Chapter IV) that the diffusion is <u>very conservative</u>, but is of the correct order of magnitude. However, the relative error drops as $B(0) + D(0) \to \infty$. Nevertheless, the use of diffusion provides insights, and simple interpretation analysis of the problem without an excessive computational burden. Such insights may then be used for more detailed investigations if required.

Analysis such as Table (5.1) suggests that sensitivity analysis can be carried out easily on the combat-logistics model, by changing logistics and/or combat parameters, and re-solving the resulting model. The computational burdens of re-solving are not at all excessive, in part because the calculation of leakage probability is facilitated by the diffusion approximations.

Note that the leakage obtained under optimal logistics assignment appear excessive, unless the C^3 capability, measured by θ, can be improved, or perhaps the early warning, and hence combat time, be extended. As an example of the type of analysis that can be carried out by the use of the CL model, a training program is analyzed below.

b. Training

Suppose the decision maker would like to investigate the effect of adopting a certain training program for the pilots. Suppose that implementation of the new training program results in improvement of pilots' combat and firing skills. Such improvement could be represented through an increase in the combat parameter α_D. Suppose that if the new training program is adopted, then the defender killing rate increases from $\alpha_D = 3.0$ to $\alpha_D' = 4.0$ bombers per hour.

Assume that aircraft are required to be operated at a higher rate (i.e., no simulators) if the program is adopted. The effect of the training program on the logistics

system can be represented through the failure rate λ. Suppose that the training program causes the failure rate to increase from $\lambda = 0.5$ to $\lambda' = 1.0$.

The effect of the new training program on the overall squadron performance can be evaluated by the use of CL. The new parameters were considered by the 1MCL model, which gave the following values: WS* = [7 15 1] with an objective value of 0.398, i.e., a probability of leakage of 0.602.

Comparing the results before and after the implementation of the training program, we find that even though the training program results in an improvement in the skills of the pilots (represented by the higher killing rate), it has a considerable effect on the logistics system which in turn increased the probability of leakage (.4712 before implementation versus .602 after implementation).

5. <u>2MCL Using I Model</u>

The following parameters are used as an illustration for the 2MCL model using the I model.

a. Cost

- The squadron has been allocated a budget of K = \$75 million for purchasing WS.
- Each aircraft costs c_0 = \$10 million.
- Each module of Type 1 costs c_1 = \$2 million.
- Each module of Type 2 costs c_2 = \$1 million.
- E_1 costs α_1 = \$10 million, while E_2 costs α_2 = \$5 million.

- A repairman for Type 1 has an annual salary of r_1 = \$20 thousand, and a repairman for Type 2 is paid and annual salary of r_2 = \$10 thousand.

- The squadron has an annual budget for salaries of K_1 = \$80 thousand for repairmen.

b. Logistics

- Each aircraft fails at a rate of $\lambda = 0.5$ aircraft per day.

- Condition probability of an aircraft failing due to Type 1 module is $p_1 = 0.4$.

- Conditional probability of an aircraft failing due to Type 2 module is $p_2 = 0.5$.

- Conditional probability of an aircraft failing due to both types of modules is $p_{12} = 0.1$.

- An individual failed module of Type 1, or Type 2, has, respectively, a repair rate $\mu_1 = 0.5$, or $\mu_2 = 1.0$, modules per day, where only one repairman (or crew) can work on a failed module at any time.

- $\rho_1 = 2$ and $\rho_2 = 2$.

c. Combat Data

- Each defender kills at a rate of $\alpha_D = 3.0$ bombers per hour.

- If E_1 is deployed then the combat period is $t_1 = 1.0$ hours, while if E_2 is deployed then the combat period is $t_2 = 0.8$ hours.

The 2MCL, in this case, becomes:

$$\max_{WS} \sum_{j=0}^{a} \Phi\left\{\frac{3-\mu(12,j,t)}{\sigma(12,j,t)}\right\} P\{D(0) = j\}$$

s.t.

$$10a + 2M_1 + M_2 + 10E_1 + 5E_2 \leq 75$$

$$2R_1 + R_2 \leq 8$$

$$E_1 + E_2 = 1$$

$$a \leq M_1 \leq 2a$$

$$a \leq M_2 \leq 2a$$

$$1 \leq R_1 \leq M_1$$

$$1 \leq R_2 \leq M_2$$

$$a \geq 2$$

E_1, E_2 binary

a, M_1, M_2, R_1 and R_2 are positive integers

2MCL(1) is solved iteratively over all possible WS that satisfy the constraints. For every j, $\mu(12,j,t)$ and $\sigma^2(12,j,t)$ are determined using Result (4.3) of Chapter IV where $t = t_1E_1 + t_2E_2$. For every WS, $P\{D(0) = j\}$ is computed by solving the forward Chapman-Kolmogorov equation, using proposition (2.1). Table (5.2) shows the different WS's when E_1 is deployed and the corresponding objective value. We find that if E_1 is deployed, then the optimal WS is:

$$WS_1 = [4,8,8,3,2,1]$$

TABLE 5.2

ILLUSTRATION OF 2MCL(1) FOR I SCENARIO WITH E1 DEPLOYED

A	M1	M2	R1	R2	E1	OBJECTIVE VALUE
	WS					
2	2	4	1	4	1	0.0439
2	2	4	2	4	1	0.0487
2	2	4	3	2	1	0.0485
2	3	4	1	4	1	0.0625
2	3	4	2	4	1	0.0830
2	3	4	3	2	1	0.0836
2	4	4	1	4	1	0.0728
2	4	4	2	4	1	0.0964
2	4	4	3	2	1	0.0994
3	3	6	1	6	1	0.1399
3	3	6	2	4	1	0.1931
3	3	6	3	2	1	0.1955
3	4	6	1	6	1	0.1839
3	4	6	2	4	1	0.3160
3	4	6	3	2	1	0.3323
3	5	6	1	6	1	0.2078
3	5	6	2	4	1	0.3791
3	5	6	3	2	1	0.4147
3	6	6	1	6	1	0.2218
3	6	6	2	4	1	0.4162
3	6	6	3	2	1	0.4504
4	4	8	1	6	1	0.2086
4	4	8	2	4	1	0.3722
4	4	8	3	2	1	0.3961
4	5	8	1	6	1	0.2332
4	5	8	2	4	1	0.5027
4	5	8	3	2	1	0.5695
4	6	8	1	6	1	0.2520
4	6	8	2	4	1	0.5777
4	6	8	3	2	1	0.6791
4	7	8	1	6	1	0.2586
4	7	8	2	4	1	0.6265
4	7	8	3	2	1	0.7371
4	8	8	1	6	1	0.2619
4	8	8	2	4	1	0.6608
4	8	8	3	2	1	0.7706
5	5	5	1	6	1	0.2289
5	5	5	2	4	1	0.4648
5	5	5	3	2	1	0.5100

I.e., buy 4 aircraft, 8 modules of Type 1, 8 modules of Type 2, assign 3 repairmen to Shop 1, and 2 repairmen to Shop 2, while the last component being 1 indicates the deployment of E_1. Choosing WS_1, and assuming the 2 module logistics system together with the I scenario, results in an objective value of 0.7706.

Table (5.3) shows the different WS's when E_2 is deployed, and the corresponding objective value. We find that if E_2 is deployed, then the optimal WS is:

$$WS_2 = [4 \quad 8 \quad 8 \quad 3 \quad 2 \quad 0]$$

i.e., buy 4 aircraft, 8 modules of Type 1, 8 modules of Type 2, assign 3 repairmen to Shop 1, and 2 repairmen to Shop 2, where the last component being 0 indicates the deployment of E_2. This yields an objective value of 0.4925.

Since WS_1 results in a higher objective value than WS_2, the optimal solution for 2MCL is:

$$WS^* = [4 \quad 8 \quad 8 \quad 3 \quad 2 \quad 1];$$

with leaking probability of 0.2192. I.e.,

$$a^* = 4$$

$$M_1^* = 8$$

$$M_2^* = 8$$

$$R_1^* = 3$$

TABLE 5.3

ILLUSTRATION OF 2MCL(1) FOR I
SCENARIO WITH E2 DEPLOYED

A	M1	WS M2	R1	R2	E1	OBJECTIVE VALUE
2	2	4	1	4	0	0.0110
2	2	4	2	4	0	0.0122
2	2	4	3	2	0	0.0121
2	3	4	1	4	0	0.0156
2	3	4	2	4	0	0.0207
2	3	4	3	2	0	0.0209
2	4	4	1	4	0	0.0182
2	4	4	2	4	0	0.0241
2	4	4	3	2	0	0.0249
3	3	6	1	6	0	0.0614
3	3	6	2	4	0	0.0848
3	3	6	3	2	0	0.0857
3	4	6	1	6	0	0.0946
3	4	6	2	4	0	0.1514
3	4	6	3	2	0	0.1590
3	5	6	1	6	0	0.0972
3	5	6	2	4	0	0.1356
3	5	6	3	2	0	0.2043
3	6	6	1	6	0	0.1045
3	6	6	2	4	0	0.2057
3	6	6	3	2	0	0.2239
4	4	8	1	6	0	0.1090
4	4	8	2	4	0	0.2028
4	4	8	3	2	0	0.2155
4	5	8	1	6	0	0.1295
4	5	8	2	4	0	0.2964
4	5	8	3	2	0	0.3401
4	6	8	1	6	0	0.1390
4	6	8	2	4	0	0.3534
4	6	8	3	2	0	0.4231
4	7	8	1	6	0	0.1436
4	7	8	2	4	0	0.3891
4	7	8	3	2	0	0.4671
4	8	8	1	6	0	0.1459
4	8	8	2	4	0	0.4142
4	8	8	3	2	0	0.4925
5	5	10	1	6	0	0.1340
5	5	10	2	4	0	0.3169
5	5	10	3	2	0	0.3661
5	6	8	1	6	0	0.1432
5	6	8	2	4	0	0.3927
5	6	8	3	2	0	0.4766
5	7	6	1	6	0	0.1444
5	7	6	2	4	0	0.4113
5	7	6	3	2	0	0.4913

$$R_2^* = 2$$

$$E_1^* = 1$$

$$E_2^* = 0$$

Solving the forward Chapman-Kolmogorov equations for the logistics model, we obtain the following readiness distribution:

j	=	0	1	2	3	4
$P\{D(0) = j\}$	=	0.0023	0.0143	0.0435	0.0879	0.8519

Calculate $P\{B(1) \leq 3 | B(0) = 12, D(0) = j\}$ for $j = 0,1,\ldots,4$, using the Markovian combat model, i.e., by solving the resulting forward Chapman-Kolmogorov equations (4.59) numerically. This gives

j	=	0	1	2	3	4
$P\{B(1) \geq 3 \mid B(0) = 12, D(0) = j\}$	=	0.0	0.0	0.15	0.54	0.79

Therefore,

$$\sum_{j=0}^{4} P\{B(1) \geq 3 | B(0) = 12, D(0) = j\} P\{D(0) = j\} = 0.73$$

i.e., a leakage probability of 0.27.

Hence, the 2MCL solution is a reasonably good approximation for the Markovian formulation with

$$\text{relative error} = \frac{|0.27 - 0.22|}{0.27} = .19$$

6. Other Cases: Sensitivity Analysis

Substituting WS* in 2MCL, we obtain:

$$10a^* + 2M_1^* + M_2^* + 10E_1^* + 5E_2^* < 75$$

$$2R_1^* + R_2^* = 8$$

$$M_1^* = 2a^*$$

$$M_2^* = 2a^*$$

Comparing the above equations for the optimal point with the set of constraints for 2MCL, we find that:

1. M_1^* and M_2^* are at their upper bounds.
2. The annual budget constraint is binding.
3. The fixed budget constraint is not binding since $75-10a^* + 2M_1^* + M_2^* + 10E_1^* + 5E_2^* = 1$, i.e., when $a = 4$, increasing M_1 and M_2 for given R_1 and R_2 results in improvement of the objective value, but M_1 and M_2 have reached their upper bound before the fixed budget constraint becomes binding.

a. Case 1

Suppose we increase the upper bound for M_2, reflected by increasing ρ_2 to 2.5 (i.e., change in the right-hand-side). The solution for the new 2MCL is shown in Table (5.4). The new optimal point becomes:

$$WS^* = [4 \quad 8 \quad 9 \quad 3 \quad 2 \quad 1]$$

TABLE 5.4

ILLUSTRATION FOR THE EFFECT OF CHANGING THE RIGHT-HAND-SIDE OF 2MCL(1) FOR I SCENARIO

		WS				OSJECTIVE
A	M1	M2	R1	R2	E1	VALUE
2	2	5	1	5	1	0.0442
2	2	5	2	4	1	0.0491
2	2	5	3	2	1	0.0490
2	3	5	1	5	1	0.0630
2	3	5	2	4	1	0.0839
2	3	5	3	2	1	0.0850
2	4	5	1	5	1	0.0735
2	4	5	2	4	1	0.0978
2	4	5	3	2	1	0.1015
3	3	7	1	6	1	0.1401
3	3	7	2	4	1	0.1936
3	3	7	3	2	1	0.1967
3	4	7	1	6	1	0.1844
3	4	7	2	4	1	0.3172
3	4	7	3	2	1	0.3360
3	5	7	1	6	1	0.2083
3	5	7	2	4	1	0.3810
3	5	7	3	2	1	0.4210
3	6	7	1	6	1	0.2223
3	6	7	2	4	1	0.4186
3	6	7	3	2	1	0.4582
4	4	10	1	6	1	0.2087
4	4	10	2	4	1	0.3724
4	4	10	3	2	1	0.3979
4	5	10	1	6	1	0.2382
4	5	10	2	4	1	0.5033
4	5	10	3	2	1	0.5754
4	6	10	1	6	1	0.2520
4	6	10	2	4	1	0.5787
4	6	10	3	2	1	0.6896
4	7	10	1	6	1	0.2587
4	7	10	2	4	1	0.6277
4	7	10	3	2	1	0.7509
4	8	9	1	6	1	0.2620
4	8	9	2	4	1	0.6619
4	8	9	3	2	1	0.7908
5	5	5	1	5	1	0.2239
5	5	5	2	4	1	0.4648
5	5	5	3	2	1	0.5100

with an objective value of 0.7808. The budget constraint is now binding, and M_1 is at its upper bound while M_2 is not.

b. Case 2

Suppose we are interested in seeing the effect of changing the repair rate, i.e., change μ_1 to 2.0 while ρ_2 is still 2.5. The solution for the resulting 2MCL is shown in Table (5.5). The new optimal point is:

$$WS^* = [4 \quad 8 \quad 9 \quad 2 \quad 4 \quad 1]$$

with an objective value of 0.8574. Note that increasing the repair rate in Shop 1 has resulted in a corresponding increase in the objective value, and caused the optimal number of repairmen to be assigned to Shop 1 drops from 3 to 2, and to Shop 2 to increase from 2 to 4.

TABLE 5.5

ILLUSTRATION FOR THE EFFECT OF CHANGING
THE REPAIR RATE AT SHOP 1 OF 2MCL(1)
FOR I SCENARIO WITH E1 DEPLOYED

A	M1	WS M2	R1	R2	E1	OBJECTIVE VALUE
2	2	5	1	5	1	0.0860
2	2	5	2	4	1	0.0871
2	2	5	3	2	1	0.0867
2	3	5	1	5	1	0.1042
2	3	5	2	4	1	0.1071
2	3	5	3	2	1	0.1065
2	4	5	1	5	1	0.1085
2	4	5	2	4	1	0.1096
2	4	5	3	2	1	0.1090
3	3	7	1	6	1	0.3654
3	3	7	2	4	1	0.3795
3	3	7	3	2	1	0.3770
3	4	7	1	6	1	0.4515
3	4	7	2	4	1	C.4770
3	4	7	3	2	1	0.4723
3	5	7	1	6	1	0.4816
3	5	7	2	4	1	0.4948
3	5	7	3	2	1	0.4897
3	6	7	1	6	1	0.4926
3	6	7	2	4	1	0.4981
3	6	7	3	2	1	0.4918
4	4	10	1	6	1	0.6493
4	4	10	2	4	1	0.6970
4	4	10	3	2	1	0.6949
4	5	10	1	6	1	0.7615
4	5	10	2	4	1	0.8196
4	5	10	3	2	1	0.8143
4	6	10	1	6	1	0.8120
4	6	10	2	4	1	0.8491
4	6	10	3	2	1	0.8422
4	7	10	1	6	1	0.8360
4	7	10	2	4	1	0.8565
4	7	10	3	2	1	0.8467
4	8	9	1	6	1	0.8472
4	8	9	2	4	1	0.8574
4	8	9	3	2	1	0.8394
5	5	5	1	5	1	0.6691
5	5	5	2	4	1	0.7083
5	5	5	3	2	1	0.6751

VI. SUMMARY AND CONCLUSION

A. GENERAL REVIEW

The objective of this research has been to construct and utilize analytical models for certain combat-logistics situations. Since combat outcome is often crucially influenced by a combatant's readiness, and readiness depends strongly upon the adequacy of logistics support, it is necessary to link an appropriate logistics model with one for combat to create a combat-logistics (CL) model. Use of such a combined model allows the effect of the organization and budgetary constraints of the logistics system upon combat outcome to be directly assessed and optimized.

The particular situation considered in this dissertation has been that of a squadron of aircraft designated to defend a small areas against surprise attack by wave of bombers. Upon detection of the bombers, ready defenders are vectored towards them and engage them in combat. Combat outcome is summarized by a measure of bomber leakage through the combat zone.

Both logistics and combat are conducted in environments of substantial random variability and the numbers on both sides are small. Consequently stochastic models have been developed for the logistics and combat phases of the scenario described, and these models have then been linked.

The linked CL model allows a decision maker to identify fixed resource (budget-constrained) allocations that effectively minimize a chosen measure of bomber leakage.

In order to provide a usable analytical tool, i.e., a CL model that can provide practical, reasonable and useful numerical outputs with feasible computational resources, mathematical approximations have been invoked. Particularly is this so in the combat modelling area, where an essentially transient and multi-state situation must be modelled. Various approaches have been taken, but most emphasis has been given to a form of diffusion approximation. The quality of this approximation improves when opposing forces become large, but the approximation provides useful accurate inputs to a logistics system optimization stage that would otherwise be computationally prohibitive.

This chapter reviews the models constructed for analyzing peacetime and wartime activities. It highlights the main results obtained, and it points out areas that require further research.

B. PEACETIME (NON-COMBAT) ACTIVITIES

1. Single-Module Logistics Model

The peacetime (prior-to-combat) part of the combat-logistics process was analyzed with logistics models in Chapters II and III. This was accomplished initially by representing a weapon system, i.e., a defending aircraft, subject to failures as a single module. For this single

module situation, a logistics model was developed to describe aircraft readiness as a function of the number of spare parts, number of aircraft, number of repair facilities and failure and repair rates. The limitation of the model is the representation of an entire aircraft as a single module.

2. Two-Module Logistics Model

To overcome the limitation of the single-module formulation a model was developed for the case in which the aircraft consists of two modules both of which are required for the aircraft to be considered mission capable. The model allows for simultaneous failures of the two modules, permitting much greater operational realism in representing the logistics activities of an aircraft squadron.

The model is a bivariate-continuous-time Markov process. The size of the corresponding infinitesimal generator is $(M_1+1)(M_2+1)$, which becomes very large for even moderate values of M_1 and M_2. In order to solve this problem, the matrix geometric approach was implemented. With this approach the infinitesimal generator of the process was constructed and the limiting distribution and the Laplace transforms for general initial conditions were determined.

3. Readiness

In order to compute the probability distribution for the number of aircraft ready to engage in combat, the

surprise phenomenon of war was considered. In this study the element of surprise was classified as total or partial surprise. Under the total surprise scenario, the defender has no time to prepare for combat, and hence there is no opportunity to cause changes to the logistics process. Under the scenario of partial surprise, the defender is assumed to have an exponential amount of time to prepare for combat. The combat preparation phase was modelled by decreasing the failure rate of an individual aircraft, and increasing the repair rate of an individual repairman. Hence, a new logistics process is introduced. This new process uses the limiting distribution of the old logistics process as an initial condition. The probability distribution of the number of aircraft ready to engage in combat at time of attack is then computed by solving a linear system of equations for the Laplace transforms. The cases studied indicate the degree with which readiness is improved when there is a warning time (i.e., partial surprise). This evaluation can be used to estimate the effect of an improved intelligence system on aircraft squadron performance.

4. Repairman Allocation Model

The problem of assigning repairmen (or crews) as a function of the state of the system was analyzed in Chapter III. Optimal assignment was determined by formulating the operational logistics system as a multivariate

discrete-state continuous-time Markov decision process. The optimal policy was then computed by applying dynamic programming and linear programming methods to the Markov decision process. In order to use Howard's (1976) solution algorithms for solving a univariate Markov decision process, a general one-to-one transformation function for mapping a multivariate state space on to a univariate state space was derived. This transformation facilitated solving the multivariate process as a univariate process using existing algorithms and theory. The results were then retransformed to the multivariate case for the final solution.

5. Further Research

The logistics models presented in this thesis assume modules that are operational and not installed on an aircraft to be stored at a supply depot in a ready-to-install status. When an aircraft lands with a failed module, replacement is carried out instantaneously if there is a module of the same type in the supply depot. This assumption may be relaxed by introducing a random module replacement time.

The models studied in this research assume that the repair shop is capable of conducting all types of repair. A multi-echelon repair system with base and depot repair may be considered explicitly by introducing a random delay during which time a module is diagnosed as repairable at the base with probability ρ, or requires depot repair with

probability 1-β. See Sherbrooke (1968), Muckstadt (1973), Hillestad (1982), and Homesby (1985).

Other distributions can be assumed for the warning time. Gaver (1966) examined a class of densities that is convenient for representing warning times and presented a method for obtaining an approximate inverse of the Laplace transform. This provides some generalization in modelling surprise and the effect of an improved intelligence system. This is an area that merits further research. The results presented in Chapter II show that the element of surprise plays a major role in providing a link between the logistics and combat processes through its impact on the initial conditions of the combat process.

The repairmen allocation model assumes neither cost nor time involved in assigning/reassigning repairmen to shops. The assumption of no elapsed time required to reassign repairmen to shops may be relaxed by introducing a "state of transition" for repairmen. If a repairman in one shop is reassigned to another, then the repairman experiences a delay (is said to be in limbo) for an exponential amount of time. One can also include costs in the repairman assignment problem. Two straightforward ways to do this are:

(i) Introduce another objective function. The model then has two objective functions. One is to maximize the long run expected number of aircraft ready to engage. The second is to minimize the long run expected cost. The model can then be solved as a

multiple-objective Markov decision process; see Viswanathan, Aggarwal and Nair (1977).

(ii) Add a new constraint which indicates that the long run cost must not exceed a certain bound. The resulting model can then be solved for different values of the bound. A plot of the objective value (i.e., the expected number of aircraft operational) versus the long run expected cost can then be constructed.

C. WARTIME (COMBAT) ACTIVITIES

Analytical models for representing air-to-air combat were studied in Chapter IV. The major areas are summarized below.

1. The Finite-Re-Engagement Rate Combat Model (the BCD Model)

The BCD model assumes that defenders are vulnerable and spend some time searching for a free enemy bomber. Once a free bomber is detected, it is then engaged by a free defender until either one of them is killed. If the bomber is killed, the defender once again searches for a free bomber; if a bomber wins, or is never engaged in combat, it may leak through the combat zone and attack the area defended.

The combat process under the BCD scenario was initially modelled deterministically by a system of differential equations. The deterministic model provides representation of the number of free bombers, bombers (hence defenders) in combat, and free defenders as a function of time. A simple criteria for defenders/bombers to win was

obtained in case combat time available is unlimited. An equation that predicts (approximately) the expected number of defenders/bombers surviving was presented.

The deterministic model was then generalized by a diffusion model. The use of diffusion theory allowed us to represent some combat measures of effectiveness in near closed form mathematical equations. This, in turn, simplified the linking of the combat models to the logistics models.

A trivariate-continuous-time discrete-state Markov model (in particular a trivariate pure death process) was developed to represent the BCD combat process. A method for computing the Laplace transforms of the time-state probabilities was presented. The Markov process was simulated to check the results of the diffusion model. It was found that the diffusion results provided good approximations to the Markov process.

The diffusion results were found to produce an approximation to the expected number of aircraft alive at time t that was identical to that provided by the deterministic models. In addition, the diffusion results provide an approximation for the variance-covariance matrix. Thus, probability statements can be e'aluated by using the normal approximation.

The dimension of the system of differential equations resulting from the diffusion approximations is not a

function of the initial size of the forces. This allows military analysts to model economically combat processes for larger size problems. It also allows military analysts to derive expressions for approximating probability statements that represent measure of effectiveness and to compute their values without expenditure of excessive computer time.

2. The Infinite Re-Engagement Rate, I Model

The I model assumes that defenders are invulnerable, and once a bomber is killed a free defender engages a free bomber, if any, instantaneously. The combat process was modelled deterministically by a differential equation. The equation was solved to provide an expression that approximates the expected number of bombers alive as a function of time.

The deterministic model was then generalized by developing a diffusion model. Expressions that represent the mean and variance of the number of bombers alive at time t was derived. It was found that the deterministic and the diffusion models gave the same representation of the mean.

A simple univariate pure death model was developed to represent the I combat process. It was found that the diffusion results provided good approximation to the pure death model. An expression for the expected number of bombers alive as a function of time was derived.

To allow more refined approximate calculations to be made for the I model, the large deviations method was

applied. A large deviations equation was derived to approximate the probability of at least z bombers alive at time t for relatively large values of t. It was found that the large deviations technique resulted in a better approximation for the measure of effectiveness than did the diffusion approximations. As yet there is no known way of applying large deviations to the BCD model. Hence the diffusion approximation has been used in the logistics optimization phase of the problem.

3. Further Research

The application of the large deviations method to multivariate stochastic combat models appears to be a ripe area for further research.

Diffusion models were used to generalize the deterministic models for air-to-air combat. It would be useful to develop similar models to generalize (approximate) the deterministic (Markovian) models for other types of combat (e.g., land, sea, ground-to-air, etc.). See Appendix B for a generalization of Lanchester's linear law; there it is shown that the diffusion mathematics recovers Lanchester's linear law and provides in addition a representation for the variance-covariance matrix of the approximating multivariate Gaussian process.

D. COMBAT-LOGISTICS MODEL

Chapter V was devoted to linking the analytical logistics models with the diffusion models of combat to

produce a combat-logistics model. The combat-logistics model is a non-linear-integer-mathematical programming model. That model determines the optimal number of aircraft, modules, and the type of early warning system to be procured, given fixed budgetary constraints. In addition, the optimal number of repairmen to be recruited (employed) is determined. The solution minimizes the probability of penetration of enemy bombers subject to budget, operational, and capacity constraints on the defenders.

1. <u>Further Research</u>

Further work remains to be done. The combat-logistics model described in this thesis allows a decision maker to evaluate the outcome of air-to-air combat, and hence provides the initial distribution of the number of bombers to be engaged by SAM forces. An analytical model for the ground-to-air combat, that can be grafted onto the proposed combat-logistics model, is also needed. A generalization to the combat-logistics model that allows for repeated enemy attacks over a short period of time is also needed.

E. CONCLUSION

The research presented in this thesis represents an effort to develop analytical models which can be used to assist decision makers in making some of the complex logistics decisions having an impact on combat effectiveness.

This work represents the first known effort to integrate the logistics and combat processes analytically.

Combat-logistics models of the type illustrated here appear to have much to recommend them. Making decisions under a peacetime environment that can considerably affect combat outcomes is a complex task. The use of combat-logistics models similar to those presented here can assist the decision maker in answering "what if" questions interactively without expenditure of excessive computer time. The models can also be used to assist the planner in determining the effects of implementing new policies, cuts (or increases) in the annual maintenance budget, reductions in the number of modules, increases in the number of aircraft, improvement of the intelligence system, modification of the early warning system, or improvement in the communication, command and control system.

APPENDIX A

NUMERICAL STUDIES IN LARGE DEVIATIONS

A. GENERAL

This appendix is designed to analyze the large deviations technique for evaluating probabilities at large values. The analysis includes: Deriving the large deviations equation; comparing it with actual values and with central limit theorem (CLT) results; and performing error analysis for the entire range of the independent variable.

The idea of the large deviations technique is to displace the distribution of the random variable toward the value of interest, which is situated in the tail of the distribution. Applying the Normal approximation to the displaced distribution gives a new, better approximation to the actual probability (Feller, 1971).

The appendix applies the large deviations technique to the compound Poisson process and renewal processes.

An approach to reduce the large deviations error is introduced by shifting the value of interest with a prescribed interval. This approach is fully illustrated for the compound Poisson process.

B. LARGE DEVIATIONS CONCEPT

Suppose $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random variables with a common distribution

$F(\cdot)$, such that $E[Y_k] = 0$ and $E[Y_k^2] = \sigma^2$, then the normalized sum $(\sum_{k=1}^{n} Y_k)/(\sigma\sqrt{n})$ has the "n" fold distribution F_n. By the CLT, F_n tends to the standard normal (Φ).

For large x, $\Phi(x)$ is close to unity, which indicates that for very large values of x, the CLT becomes empty. To overcome this problem, the large deviations (LD) technique was suggested by Esscher (1932). Also see Feller (1971); for a recent application of the technique see Mazundar and Gaver (1984).

We now derive the LD equation for $P\{X > z\}$, where X is either a fixed or a randomized sum of i.i.d. random variables.

Let $F_X(x)$ be the probability distribution function for X, and $\hat{F}_X(s)$ is the corresponding moment generating function; then,

$$\hat{F}_X(s) = \int_{-\infty}^{\infty} e^{su} F_X\{du\} \qquad s > 0 \tag{A-1}$$

Let V be the probability distribution associated with $F_X(\cdot)$, with mean μ and variance σ^2. Then:

$$V\{dx\} = \frac{e^{sx} F_X\{dx\}}{\hat{F}_X(s)} \tag{A-2}$$

Let $\hat{V}(\xi)$ be the moment generating function of V. Then:

$$\hat{V}(\xi) = \int_{-\infty}^{\infty} e^{\xi x} V\{dx\}$$

Substituting (A-2) and (A-1), we obtain:

$$\hat{V}(\xi) = \frac{F_X(s+\xi)}{\hat{F}_X(s)} \qquad \text{(A-3)}$$

If $K(\xi)$ is the cumulant generating function for V, then,

$$K(\xi) = \ln \hat{V}(\xi)$$

It follows that,

$$\mu(s) = K'(0)$$

and

$$\sigma^2(s) = K''(0) \ .$$

Note that μ and σ^2 are functions of s, i.e., there is a family of associated distributions for $F_X(\cdot)$.

Let $\mu(s) = z$ (i.e., center the associated distribution at the point of interest), and solve for s. Let $\tilde{s}$ be the value of s such that $\mu(s) = z$. Thus, a unique associated distribution V has been determined, i.e.:

$$V\{dx\} = \frac{e^{\tilde{s}x} F_X\{dx\}}{\hat{F}_X(\tilde{s})}$$

with:

$$\text{mean} = \mu(\tilde{s}) = z \ ,$$

and

$$\text{variance} = \sigma^2(\tilde{s}) \ .$$

Thus,

$$F_X\{dx\} = \hat{F}_X(\tilde{s}) \ e^{-\tilde{s}x} \ V\{dx\}$$

i.e.,

$$1 - F_X(z) = \hat{F}_X(\tilde{s}) \int_z^\infty e^{-\tilde{s}x} \ V\{dx\}$$

By the normal approximation:

$$1 - F_X(z) \simeq \hat{F}_X(\tilde{s}) \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\tilde{s}x} \ e^{-\frac{(x-\mu(\tilde{s}))^2}{2\sigma(\tilde{s})}} \frac{dx}{\sigma(\tilde{s})}$$

By standard substitution, and after completing the square, we get:

$$1-F_X(z) \simeq \hat{F}_X(\tilde{s})e^{-\tilde{s}\mu(\tilde{s})+\frac{\tilde{s}^2}{2}\sigma^2(\tilde{s})} \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(\omega+\tilde{s}\sigma(\tilde{s}))^2} d\omega$$

Considering the transformation $x = \omega + \tilde{s}\sigma(\tilde{s})$, we obtain:

$$1-F_X(z) \simeq \hat{F}_X(\tilde{s})e^{-\tilde{s}\mu(\tilde{s})+\frac{\tilde{s}^2}{2}\sigma^2(\tilde{s})} \frac{1}{\sqrt{2\pi}}\int_{\tilde{s}\sigma(\tilde{s})}^\infty e^{-x^2/2}\, dx$$

Hence, the LD equation is:

$$1-F_X(z) \simeq \hat{F}_X(\tilde{s})e^{-\tilde{s}\mu(\tilde{s})+\frac{\tilde{s}^2}{2}\sigma^2(\tilde{s})} [1-\Phi(\tilde{s}\sigma(\tilde{s}))] \qquad \text{(A-4)}$$

where:

$\hat{F}_X(\tilde{s})$ = *moment generating function of X*

$\mu(\tilde{s})$ = mean of V = z

$\sigma^2(\tilde{s})$ = variance of v.

The above equation is the LD equation with parameters $\{\hat{F}_X(\tilde{s}), \mu(\tilde{s}), \sigma^2(\tilde{s})\}$. The claim in this appendix is that any probability value computed for a point that is far in the tail using equation (A-4) will result in a better approximation for the actual probability than CLT.

Numerical studies were conducted to compare the CLT to the LD approximations. This is done according to:

- The length of the interval on the real line over which better results were achieved.

- The variation in the relative error over the range of the independent variable.
- The behavior of each approximation at both relatively small and large values.

C. LARGE-DEVIATIONS APPLICATIONS

1. An Application To Compound Poisson Process

Let $\{N(t);\ t \geq 0\}$ be a Poisson process with mean λt, and $\{Y_i,\ i = 1,2,\ldots\}$ be a family of i.i.d. random variables with distribution $F_Y(\cdot)$; where $N(t)$ and Y_i are assumed to be independent. Define,

$$X(t) = \begin{cases} 0 & \text{if } N(t) = 0 \\ \sum_{i=1}^{N(t)} Y_i & \text{if } N(t) > 0 \end{cases}$$

It follows that $X(t)$ is a compound Poisson process, with distribution $F_{X(t)}(\cdot)$.

Suppose we are interested in evaluating $P\{X(t) > z\}$ where z lies on the extreme right tail of the distribution.

Let $\phi_Y(s)$ be the moment generating function of Y_i, then:

$$\phi_Y(s) = \int_{-\infty}^{\infty} e^{sy}\, dF_Y\{y\} \qquad s > 0$$

Let $\hat{F}_{X(t)}(s)$ be the moment generating function of $X(t)$. Then:

$$\hat{F}_{X(t)}(s) = E[e^{sX(t)}] = E[E[e^{s\sum_{i=1}^{N(t)} Y_i} | N(t)]]$$

$$= E[(\phi_Y(s))^{N(t)}]$$

Hence,

$$\hat{F}_{X(t)}(s) = e^{-\lambda t[1-\phi_Y(s)]} \qquad s > 0 \tag{A-5}$$

To shift the distribution $F(\cdot)$ to the right, we associate with it the new probability distribution $V(\cdot)$, such that

$$V\{dx\} = \frac{e^{sx} F_{X(t)}\{dx\}}{\hat{F}_{X(t)}(s)} \qquad s > 0 \tag{A-6}$$

Let $\hat{V}(\xi)$ be the moment generating function for $V(\cdot)$. Therefore $\hat{V}(\xi)$ is given by (A-3); substituting (A-5) we obtain:

$$\hat{V}(\xi) = e^{\lambda t[\phi_Y(s+\xi)-\phi_Y(s)]} \qquad s > 0,\ \xi > 0 \tag{A-7}$$

The cumulant generating function becomes:

$$K(\xi) = \lambda t(\phi_Y(s+\xi) - \phi_Y(s)) \tag{A-8}$$

Differentiating (A-8) with respect to ξ, and setting $\xi = 0$, we obtain the following expression for the mean of v:

$$\mu(s) = \lambda t \, \phi_Y'(s)$$

Differentiating (A-8) twice with respect to ξ, and setting $\xi = 0$, we obtain the following expression for the variance of V:

$$\sigma^2(s) = \lambda t \, \phi_Y''(s)$$

Let $\tilde{s} = s(z)$: $z = \lambda t \, \phi_Y'(s(z))$, i.e., let $\tilde{s}$ be the value of s that centers the distribution at the point of interest. It follows that:

$$\mu(\tilde{s}) = \lambda t \phi_Y'(\tilde{s}) = z \qquad \text{(A-9)}$$

and

$$\sigma^2(\tilde{s}) = \lambda t \phi_Y''(\tilde{s}) \ . \qquad \text{(A-10)}$$

From (A-5), we also obtain:

$$\hat{F}_{X(t)}(\tilde{s}) = e^{-\lambda t[1-\phi_Y(\tilde{s})]} \qquad \text{(A-11)}$$

Substituting (A-9)-(A-11) into (A-4), we obtain:

$$P\{X(t) > z\} \approx e^{-\lambda t[1-\phi_Y(\tilde{s})+\tilde{s}\phi_Y'(\tilde{s})-\frac{\tilde{s}^2}{2}\phi_Y''(\tilde{s})]}$$

$$\times \; [1 - \Phi(\tilde{s}\sqrt{\lambda t \phi_Y''(\tilde{s})})] \qquad \text{(A-12)}$$

2. Example

Suppose $\{Y_i, i = 1,2,\ldots\}$ is a family of independent exponentially distributed random variables with mean μ^{-1}. Recall that $N(t)$ is a Poisson process with mean λt.

a. Computing Actual Probability

Let $f_{X(t)}$ be the density function for $X(t)$.

Then:

$$f_{X(t)}(x) = \sum_{n=1}^{\infty} P\{N(t) = n\} \, f^{n*}$$

where:

f^{n*} is the Gamma distribution with parameter (n,μ)

$$F_{X(t)}(x) = \int_0^x \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mu e^{-\mu x} \frac{(\mu x)^{n-1}}{(n-1)!} \, dx$$

$$N(t) > 0$$

$$P\{X(t) = 0\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Hence:

$$F_{X(t)}(x) = \begin{cases} e^{-\lambda t} & x = 0 \\ \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\left(1 - \sum_{j=0}^{n-1} e^{-\mu x}\frac{(\mu x)^j}{j!}\right) & x > 0 \end{cases}$$

b. Computing Probability Using the Central Limit Theorem

$$E[X(t)] = E[N(t)]\cdot E[Y] = \frac{\lambda t}{\mu}$$

$$Var(X(t)) = E[N(t)]\cdot E[Y^2] = \lambda t\left(\frac{1}{\mu^2} + \frac{1}{\mu^2}\right) = \frac{2\lambda t}{\mu^2}$$

Hence

$$P\{X(t) > z\} \overset{C.L.T.}{\simeq} 1 - \Phi\left\{\frac{z - \lambda t/\mu}{1/\mu(\sqrt{2\lambda t})}\right\} .$$

c. Computing Probability Using Large Deviations

The moment generating function for $\{Y_i, i = 1,2,\ldots\}$ is given by:

$$\phi_Y(s) = \frac{\mu}{\mu - s} \qquad s < \mu .$$

From (A-9) we get:

$$z = \lambda t \frac{\mu}{(\mu-\tilde{s})^2}$$

It follows that:

$$\tilde{s} = \mu \pm \sqrt{(\lambda t \mu)/z} \ ,$$

But $\tilde{s} < \mu$, therefore we obtain:

$$\tilde{s} = \mu - \sqrt{(\lambda t \mu)/z}$$

Hence:

$$P\{X(t) > z\} = e^{-\lambda t[1-\phi_Y(\tilde{s})+\tilde{s}\phi'_Y(\tilde{s})-\frac{\tilde{s}^2}{2}\phi''_Y(\tilde{s})]}$$

$$\times \{1 - \Phi(\tilde{s}\sqrt{\lambda t \phi''_Y(\tilde{s})})\}$$

where:

$$\tilde{s} = \mu - \sqrt{(\lambda t \mu)/z}$$

$$\phi_Y(\tilde{s}) = \frac{\mu}{\mu - \tilde{s}}$$

$$\phi'_Y(\tilde{s}) = \frac{\mu}{(\mu-\tilde{s})^2}$$

$$\phi''_Y(\tilde{s}) = \frac{2\mu}{(\mu-\tilde{s})^3}$$

d. Data

The following data are used to compare CLT and LD to the actual probabilities of the compound Poisson process:

- $\{N(t);\ t \geq 0\}$ is a Poisson process with mean λt, where $\lambda = 1, 2, 3,$ and 4.

- $\{Y_i;\ i = 1, 2, \ldots\}$ is a family of i.i.d. exponential random variables with rate $\mu = 1,\ 0.5,\ 0.33,$ and 0.25.

- The precision used for calculating the actual probability value is 0.1×10^{-6}.

- The probabilities were evaluated at "z" values ranging from 1.0 up to z_{max}, where the actual probability of being greater than z_{max} is less than 0.1×10^{-6}.

- The error for both CLT and LD is defined as:

$$\text{Relative error} = \frac{|\text{Actual Probability} - \text{Estimated Probability}|}{\text{Actual Probability}}$$

D. RESULTS ANALYSIS

Figure (A.1) shows the intervals of z over which the CLT, or the LD, gives better approximation. The figure shows 16 cases where z was chosen such that it is greater than the mean of the compound Poisson process. The figure shows that the LD technique yields better approximation for larger intervals than the CLT approximation.

Figure (A.2) shows the variation of the error with respect to z values for the first four cases, i.e., $\lambda = 1.0$, and $\mu = 1.0,\ 0.5,\ 0.33,$ and 0.25. The four cases show that errors tend to behave in a similar way. It is evident from the figure that CLT tends to lose accuracy very rapidly for values of z that are not very far from the mean value. On the other hand, it seems that the LD approximation is improving in accuracy as the value of z increases. The figure

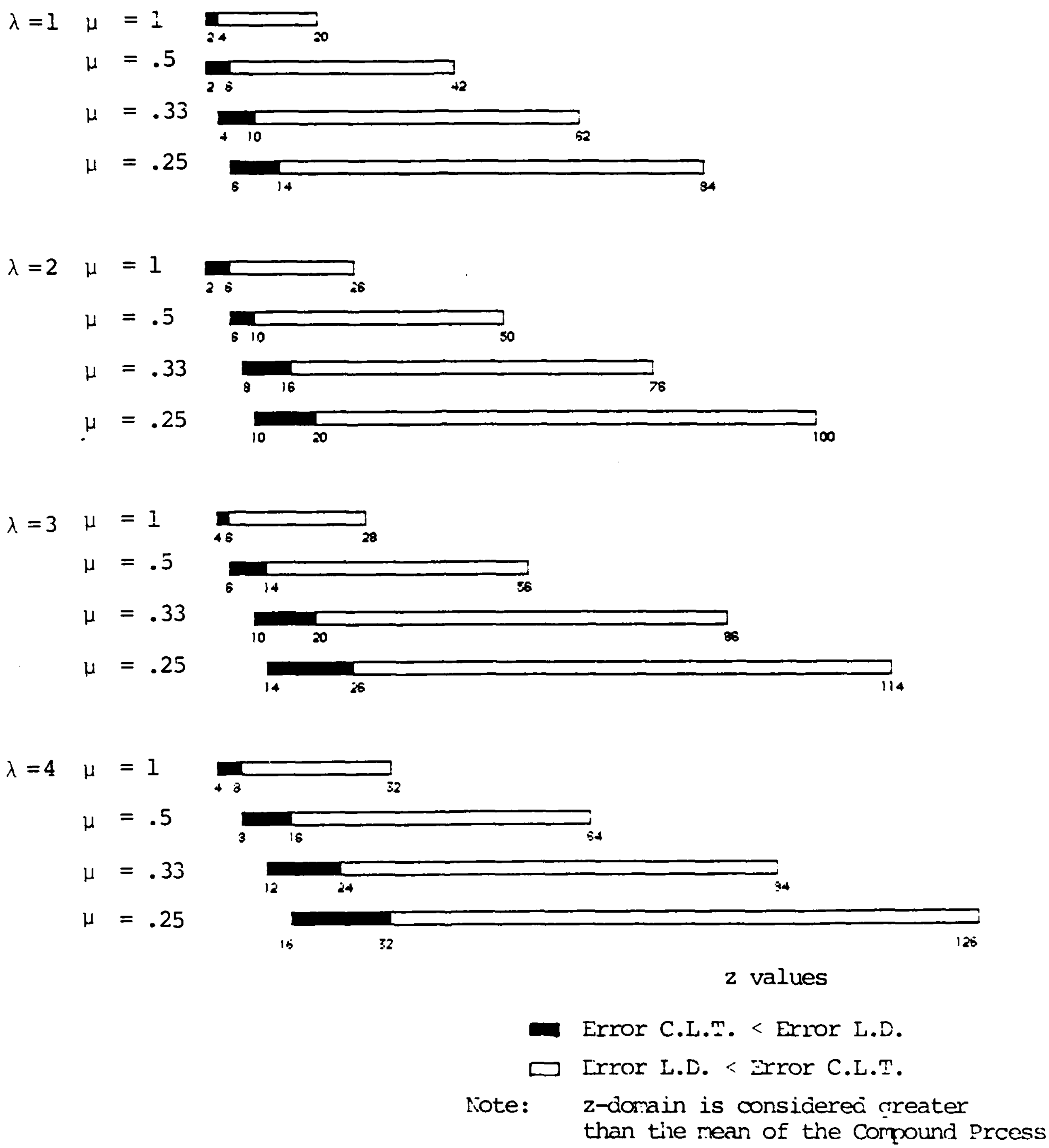

Figure A.1. Error Comparison in Z-domain: Compound Poisson Process

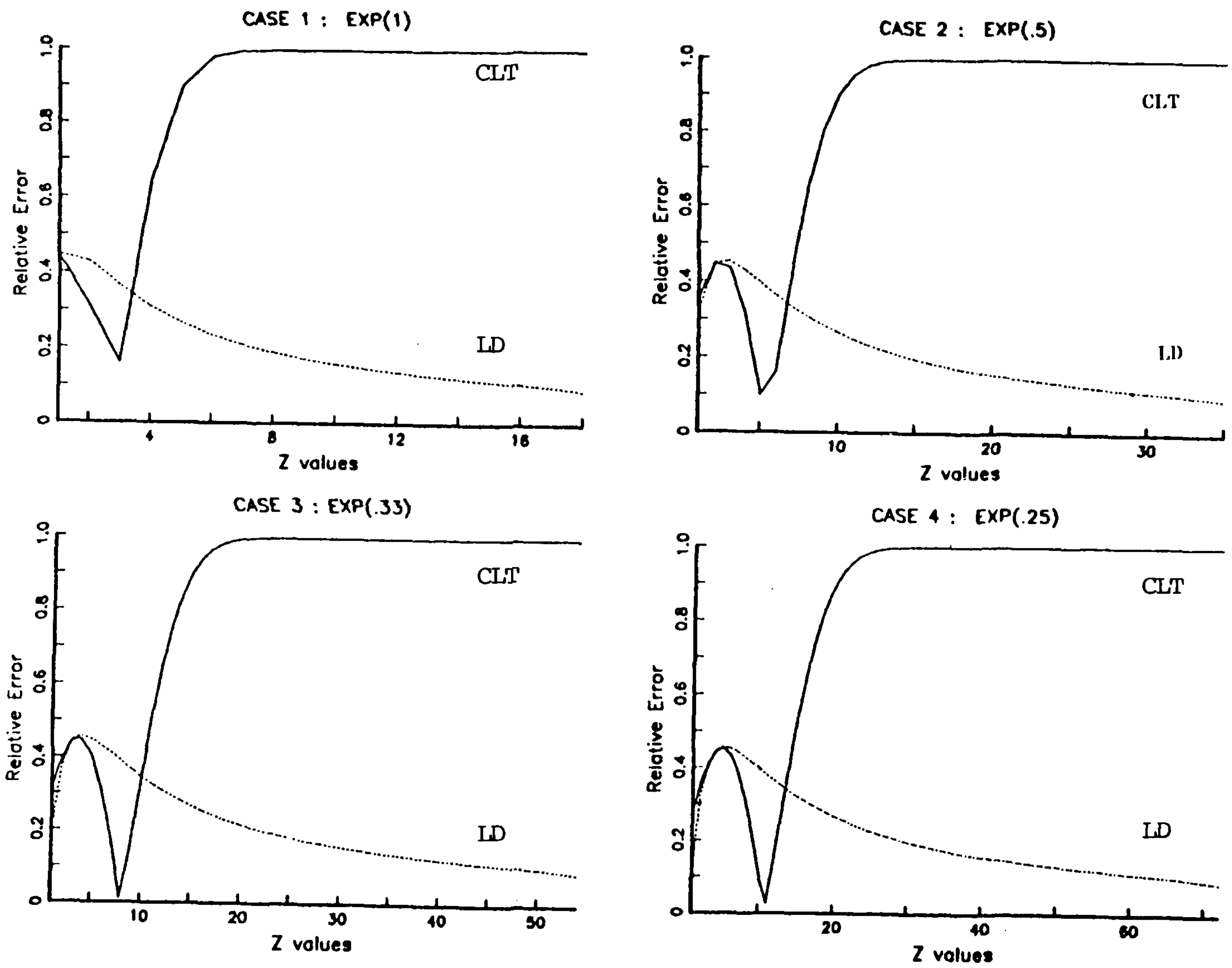

Figure A.2. Error Analysis--Compound P.P. (Poisson(1))

shows that the worst accuracy the LD gives is at relatively small values of z; the error is relatively small when compared with the maximum error the CLT generates at relatively high values of z, which in all of the cases studied was found to converge to 100% error.

Figure (A.3) illustrates the actual probabilities versus CLT and LD approximations (in logarithmic scale) for the above four cases. The plots show that generally the LD results in a good approximation, especially for large z values, and it always overestimates the actual probability (at least for the cases examined so far). The deviation from the actual probability tends to decrease as z increases. The figure indicates that LD error tends to decrease as μ increases.

To get a better understanding on the behavior of both approximations, case (4) (i.e., $\lambda = 1.0$, and $\mu = 0.25$) was studied. Figure (A.4) illustrates the comparison between the actual, CLT, and LD probabilities at small z values. The figure indicates that CLT results in a better approximation than the LD technique.

Figure (A.5) illustrates the comparison between the actual, CLT, and LD at relatively large values of z for case 4. The plot supports the previous conclusion that LD gives a good approximation for the actual probability at large values of z.

1. <u>An Application to a Renewal Process</u>

Let $\{N(t), t \geq 0\}$ be a renewal process. Suppose we are interested in evaluating $P\{N(t) < z\}$ for large values of

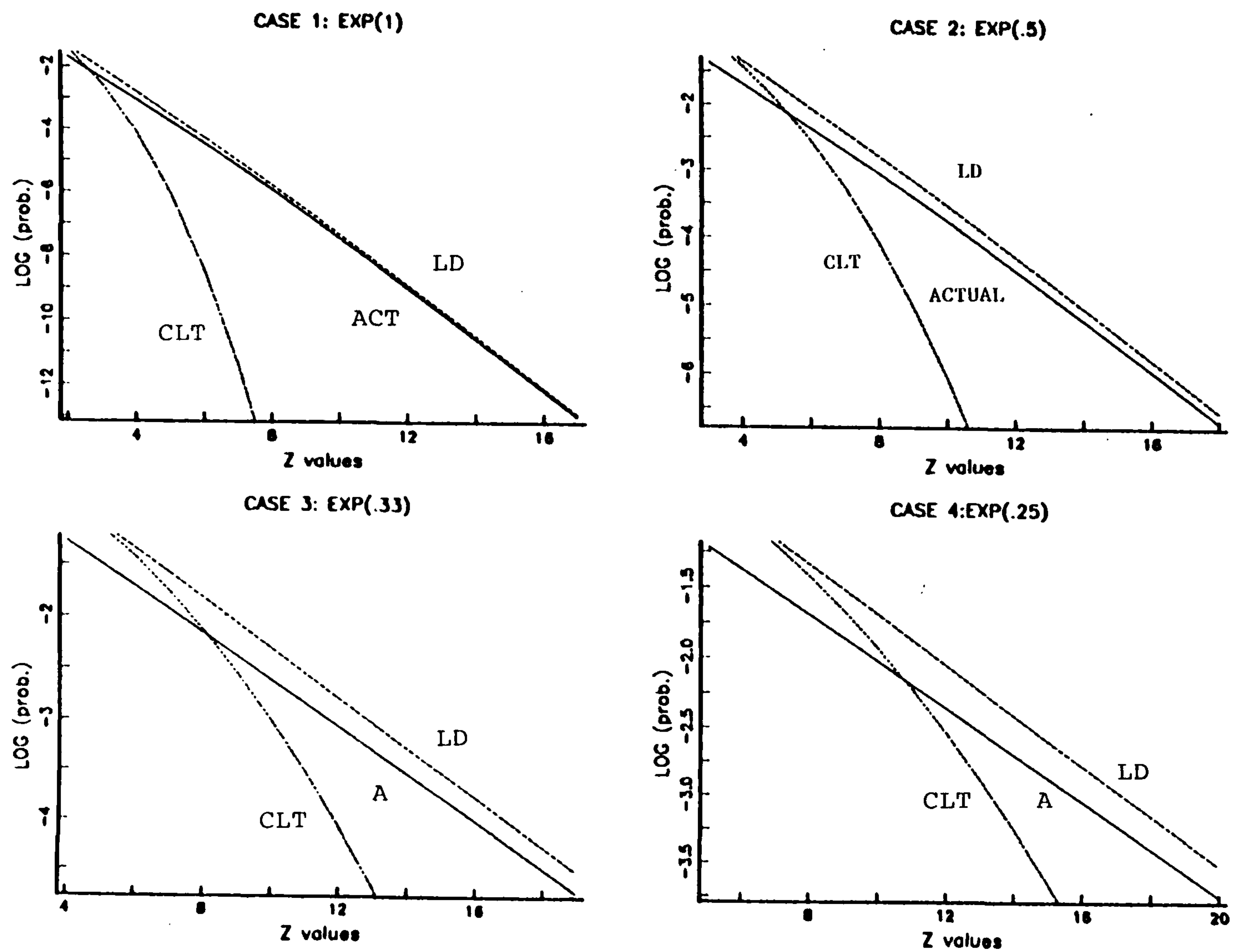

Figure A.3. Actual Prob. vs. C.L.T. and L. Dev. Prob., Compound Poisson Process Poisson (1)

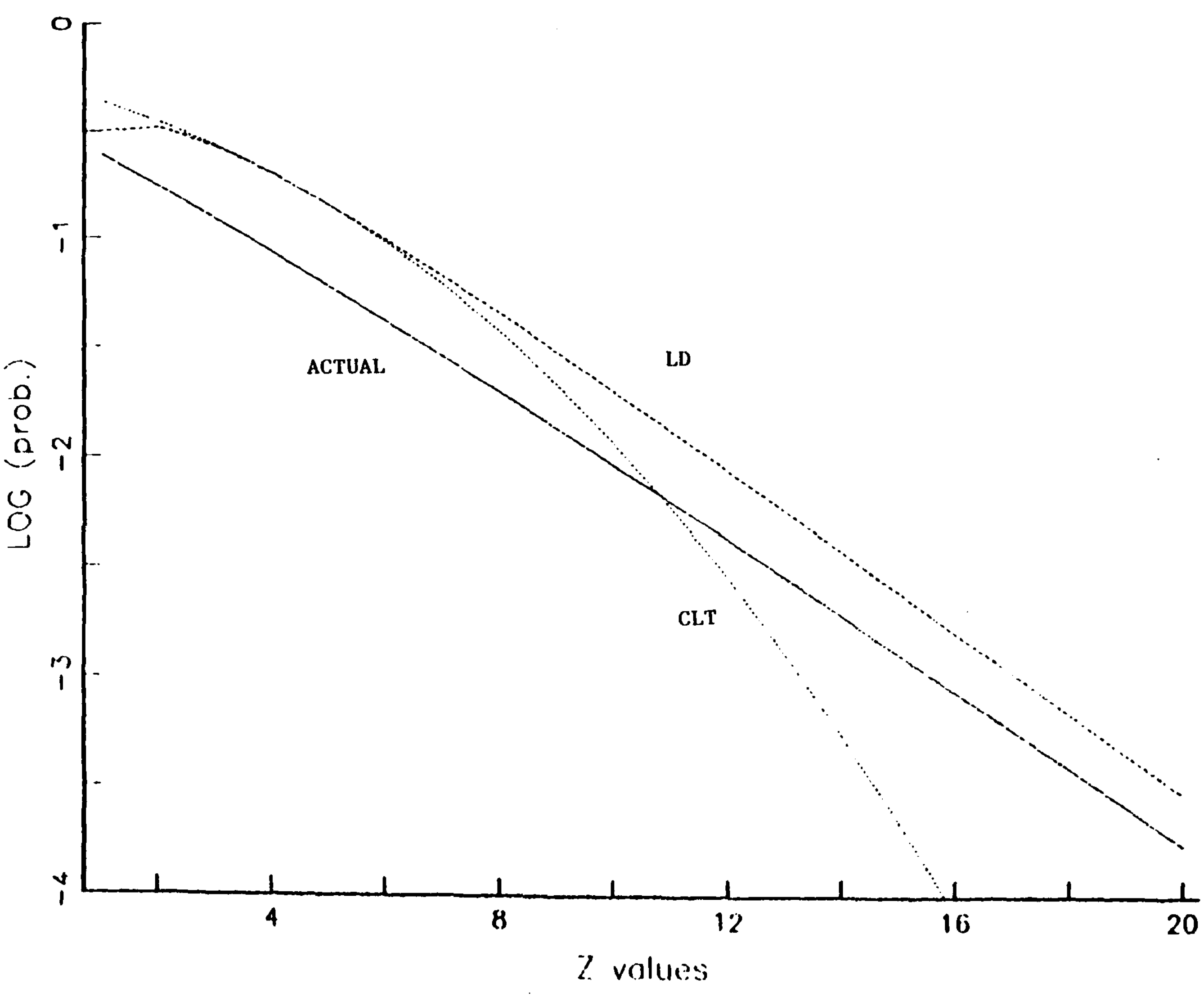

Figure A.4. Case 4: Comparison at Small Values of z

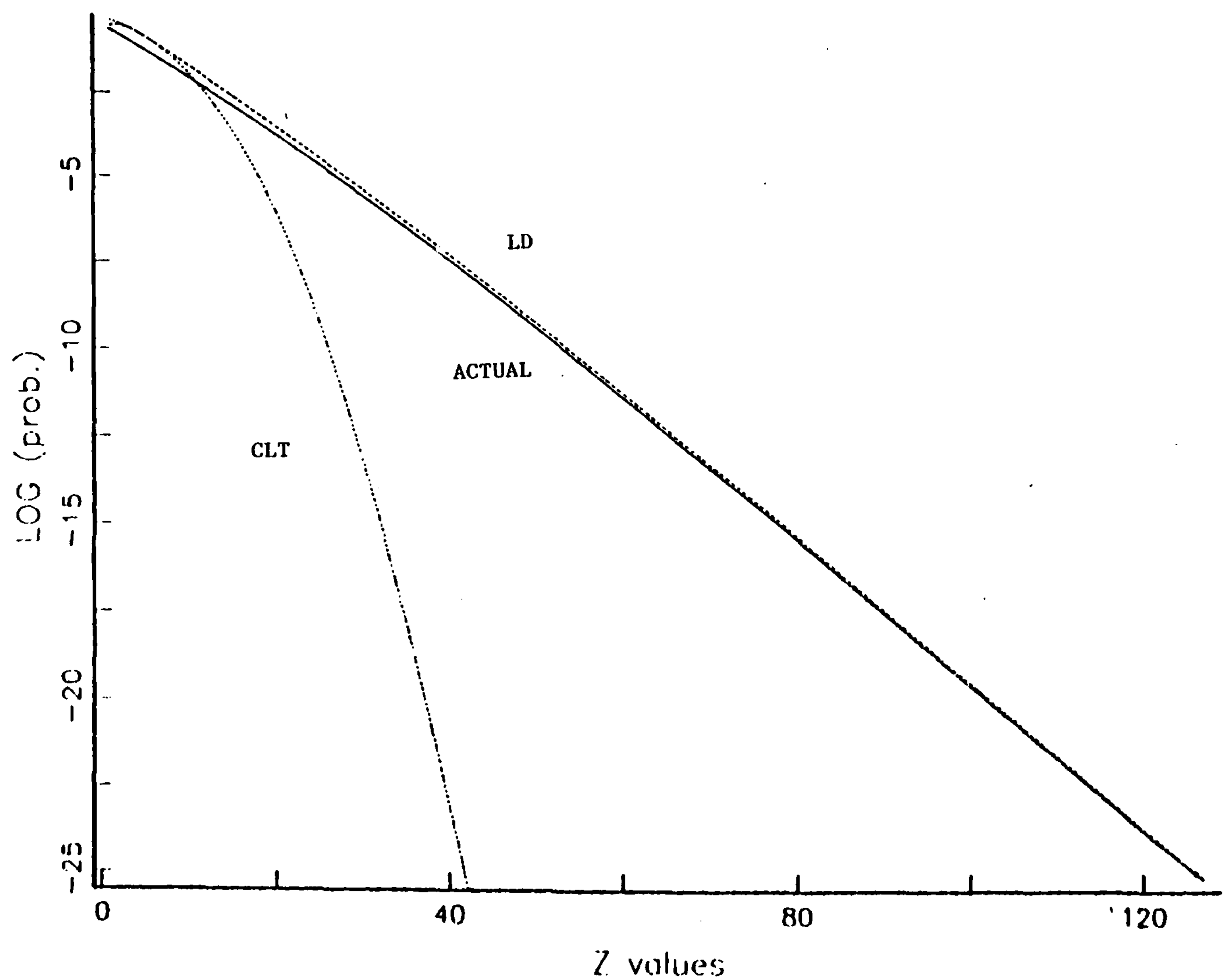

Figure A.5. Case 4: Comparison at Large Values of z

t. Define $X(n)$ to be the waiting time for the n^{th} renewal. The event $\{N(t) < z\}$ is equivalent to the event $\{X(z) > t\}$. Hence, we are interested in evaluating $P\{X(z) > t\}$.

Let Y_n be the time between the n^{th} and $(n+1)^{st}$ event, and $\{Y_n, n = 1,2,...\}$ is a family of i.i.d. random variables with distribution $F(\cdot)$. Therefore:

$$P\{X(z) > t\} = P\{\sum_{i=1}^{z} Y_i > t\}$$

where t lies in the extreme right tail of the distribution.

Let

$$\phi_Y(s) = E[e^{sY}] = \int_{-\infty}^{\infty} e^{sy}\, dF\{y\}$$

Let

$$\hat{F}_{X(n)}(s) = E[e^{sX(n)}]$$

It follows that,

$$\hat{F}_{X(n)}(s) = (\phi_Y(s))^z \qquad s > 0 \tag{A-13}$$

Define:

$$v\{dx\} = \frac{e^{sx}\, F_{X(n)}\{dx\}}{\hat{F}_{X(n)}(s)} \qquad s > 0 ,$$

The moment generating function is given by:

$$\hat{V}(\xi) = \frac{\hat{F}_{X(n)}(s+\xi)}{\hat{F}_{X(n)}(s)} \qquad s > 0, \quad \xi > 0$$

Substituting (A-13), we obtain:

$$\hat{V}(\xi) = \left(\frac{\phi_Y(s+\xi)}{\phi_Y(s)}\right)^z \tag{A-14}$$

The cumulant generating function becomes:

$$K(\xi) = z\{\ell n(\phi_Y(s+\xi)) - \ell n(\phi_Y(s))\} \tag{A-15}$$

Differentiaing (A-15) with respect to ξ, and at $\xi = 0$, we obtain:

$$\mu(s) = \frac{z \ \phi_Y'(s)}{\phi_Y(s)} \qquad s > 0$$

Differentiating (A-15) twice with respect to ξ, and at $\xi = 0$, we get:

$$\sigma^2(s) = \frac{z\phi_Y(s)\phi_Y''(s) - z(\phi_Y'(s))^2}{(\phi_Y(s))^2} \qquad s > 0$$

Let

$$\tilde{s} = s(t): \qquad t = \frac{z\phi_Y'(s)}{\phi_Y(s)} \tag{A-16}$$

i.e., $\tilde{s}$ is the value of s that centers the associated distribution at the point t.

From (A-4), the LD equation for the renewal process becomes:

$$P\{N(t) < z\} \simeq (\phi_Y(\tilde{s}))^z \, e^{-\tilde{s}t+\frac{1}{2}\tilde{s}^2\sigma^2(\tilde{s})} \{1 - \Phi(\tilde{s}\sigma(\tilde{s}))\} \qquad \text{(A-17)}$$

where:

$$\sigma^2(\tilde{s}) = \frac{z\phi_Y(\tilde{s})\phi_Y''(\tilde{s}) - z(\phi_Y'(\tilde{s}))^2}{(\phi_Y(\tilde{s}))^2},$$

and $\tilde{s}$ is given by (A-16).

The application of the CLT to the renewal process is given by the following theorem (Ross, 1983):

<u>Theorem</u>:

Let μ and σ^2, assumed finite, represent the mean and variance of an interarrival time. Then,

$$P\left\{\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right\} \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\,dx \quad \text{as} \quad t \rightarrow \infty$$

where $\{N(t), t \geq 0\}$ is a renewal process. I.e., N(t) is asymptotically normally distributed as $t \rightarrow \infty$.

E. TOWARDS A BETTER APPROXIMATION FOR LARGE DEVIATIONS

Although the LD yields a better approximation to the actual probability values, a more detailed research work was done towards improving the approximation. This work is based on investigating the skewness of the associated distribution V. The experiments done for the compound Poisson process indicates that the skewness value for the associated distribution is generally positive, i.e., it is skewed to the right. This led to carrying out some numerical studies to determine the effect of shifting the mean of the associated distribution an amount Δ.

Equation (A-9) becomes:

$$z + \Delta = \lambda t \, \phi_Y'(\tilde{s}) \tag{A-18}$$

Using (A-5) and (A-6), we obtain:

$$P\{X(t) > z\} = e^{-\lambda t[1-\phi_Y'(\tilde{s})]} \int_z^{\infty} e^{-\tilde{s}x} V\{dx\}.$$

By the Normal approximation:

$$P\{X(t) > z\} \simeq e^{-\lambda t[1-\phi_Y(\tilde{s})]} \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\tilde{s}x} e^{-\frac{(x-\lambda t\phi_Y'(\tilde{s}))^2}{2\lambda t\phi_Y''(\tilde{s})}} \frac{dx}{\sqrt{\lambda t\phi_Y''(\tilde{s})}}$$

By standard substitution (considering (A-18), we obtain:

$$P\{X(t) > z\} \simeq e^{-\lambda t[1-\phi_Y(\tilde{s})]} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\Delta}{\sqrt{\lambda t \phi_Y''(\tilde{s})}}}^{\infty} e^{-\tilde{s}(\lambda t \phi_Y'(\tilde{s}) + \omega\sqrt{\lambda t \phi_Y''(\tilde{s})} - \frac{\omega^2}{2}} d\omega$$

Completing the squares, we obtain:

$$P\{X(t) > z\} \simeq e^{-\lambda t[1-\phi_Y(\tilde{s}) + \tilde{s}\phi_Y'(\tilde{s}) - \frac{\tilde{s}^2}{2}\phi_Y''(\tilde{s})]} \frac{1}{\sqrt{2\pi}}$$

$$\times \int_{\frac{-\Delta}{\sqrt{\lambda t \phi''(\tilde{s})}}}^{\infty} e^{-\frac{1}{2}(\omega + \tilde{s}\sqrt{\lambda t \phi_Y''(\tilde{s})})^2} d\omega$$

Considering the transformation $x = \omega + \tilde{s}\sqrt{\lambda t \phi_Y''(\tilde{s})}$, we obtain:

$$P\{X(t) > z\} \simeq e^{-\lambda t[1-\phi_Y(\tilde{s}) + \tilde{s}\phi_Y'(\tilde{s}) - \frac{\tilde{s}^2}{2}\phi_Y''(\tilde{s})]} \{\Phi(\frac{\Delta}{\sqrt{\lambda t \phi_Y''(\tilde{s})}} - \tilde{s}\sqrt{\lambda t \phi_Y''(\tilde{s})})$$

The following algorithm was used for computing Δ^*, where the minimum relative error is achieved:

1. Set $\Delta_1 = 0$, $k = 1$.
2. Set the mean of the associated distribution, $\mu(\tilde{s}) = z + \Delta_k$.
3. Compute $P\{X(t) > z\}$, (actual and LD) for the compound Poisson process. Evaluate the relative error e_k, and the skewness coefficient C_k, where:

$$e_k = \frac{|\text{Actual} - \text{LD}|}{\text{Actual}},$$

and

$$C_k = \frac{K'''(0)}{\sigma^2(\tilde{s})}$$

4. If $(e_k > e_{k-1})$ stop, $\Delta^* = \Delta_{k-1}$, $k > 1$,
   Else,

   $$\Delta_{k+1} = \Delta_k + \text{Increment}$$
   $k = k+1$, go to 3.

F. DELTA-TECHNIQUE: RESULTS ANALYSIS

Implementing the proposed technique leads to a better approximation to the actual probability values. For example, it was found that for the case of $\lambda = 1$, and $\mu = 1$, the maximum relative error for z values greater than the mean of the compound Poisson process is .0006.

The numerical studies done indicate that the large deviation using the delta approach gives better approximation over the entire range of z-domain, even in the intervals where the CLT previously gave better approximations, i.e., for relatively small z values.

The numerical studies also indicate that as Δ increases, both the skewness coefficient and the relative error tend to decrease for a fixed z value. This continues until Δ^*, where the relative error starts to increase, while the skewness coefficient continues to decrease. All cases studied indicate that Δ^* is reached before the skewness coefficient reaches

the value of zero. This observation leads to the conclusion that the best associated distribution doesn't necessarily have to be symmetric.

Figure (A.6) shows the values of Δ^* for a given value of z for 16 cases (μ = 1, .5, .33, .25; and λ = 1, 2, 3, 4). The figure points out an interesting feature: The value of Δ^* can be chosen to be a fixed value for each μ. These average Δ^* values are:

If $\mu^{-1} = 1$ the average $\Delta^* = 2.2$

If $\mu^{-1} = 2$ the average $\Delta^* = 4.4$

If $\mu^{-1} = 3$ the average $\Delta^* = 6.6$

If $\mu^{-1} = 4$ the average $\Delta^* = 8.8$

Note that average $\Delta^* = 2.2(\mu^{-1})$, and it is independent of λ. Applying this criterion for determining Δ to all of the 16 cases resulted in a maximum relative error of .08 over the entire range of z. The proposed approach (i.e., $\Delta^* = 2.2(\mu^{-1})$), in addition to providing better approximations than the CLT over the entire domain, gives a simple algorithm to follow. On the other hand, if a better approximation is required, then some work has to be invested in constructing a table that indicates the value of Δ for every z for various rate values.

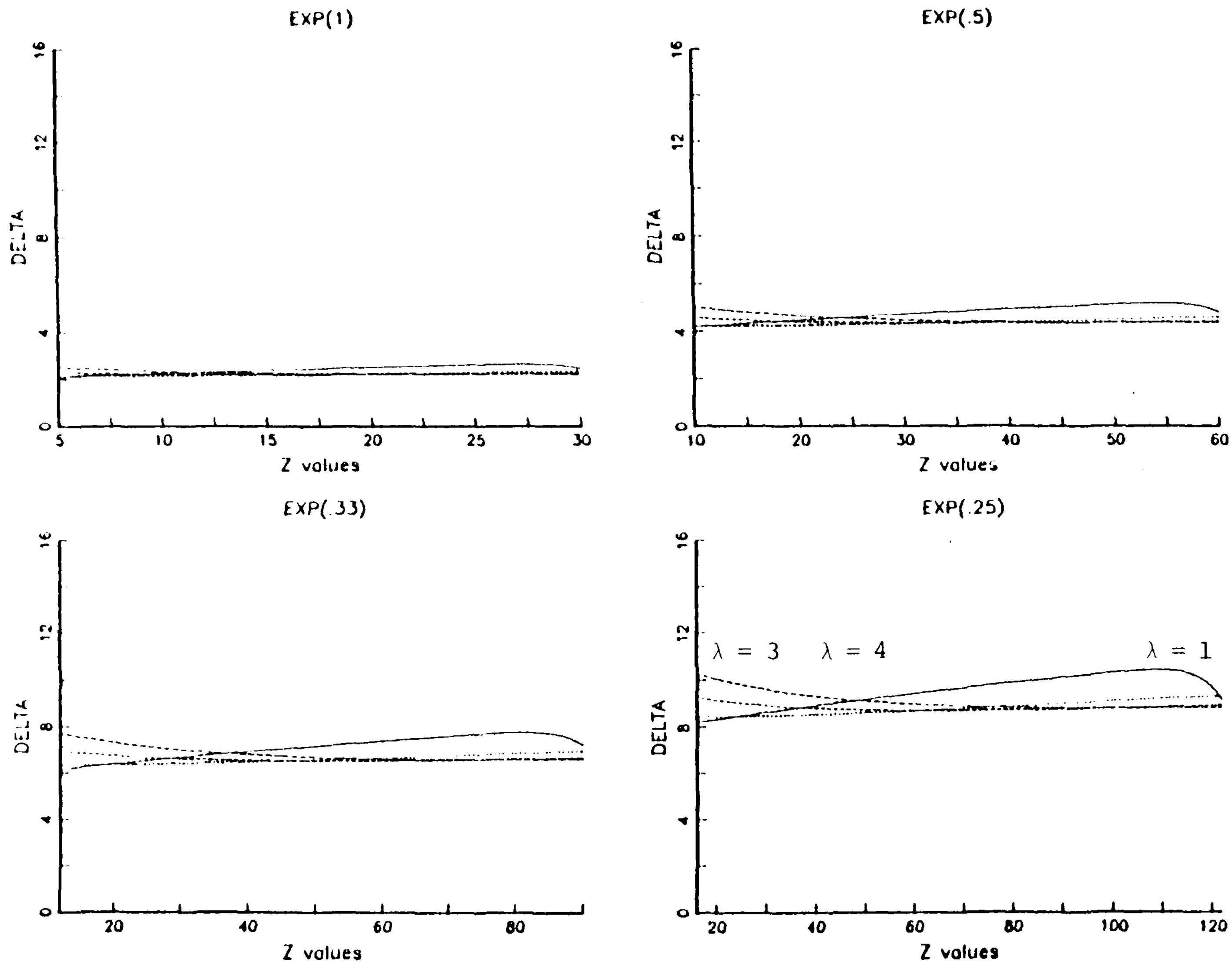

Figure A.6. Analysis of Optimum Delta Values

APPENDIX B

LANCHESTER'S LINEAR LAW: A GENERALIZATION

A. GENERAL

This appendix is designed to generalize Lanchester's linear law (LLL). Let B(t) and D(t) be the surviving combatants at time t in the attacker and defender forces, respectively. Assume B(0) = B, and D(0) = D.

Let α_D (α_B) be the rate at which an individual defender (attacker) kills an individual attacker (defender).

Define the following functions:

- b(t), an approximate representation of the number of attackers alive at time t, given the number of attackers and defenders entering combat initially.
- d(t), an approximate representation of the number of defenders alive at time t, given the number of attackers and defenders entering combat initially.

B. DETERMINISTIC MODEL: LLL MODEL

Assume that the attrition rates are proportional to the number of targets as well as the number of firers. The combat process can be modelled deterministically (LLL; see Taylor (1983)) by the following system of differential equations:

$$\frac{db(t)}{dt} = -\alpha_D\, b(t)\, d(t)$$
$$\frac{dd(t)}{dt} = -\alpha_B\, b(t)\, d(t) \qquad \text{(B.1)}$$

with initial conditions:

$$b(0) = B(0)$$

$$d(0) = D(0)$$

The first equation of (B.1) represents the rate of change of b(t) as a function of time. The expression states that the rate of change of the attackers depends on the number of attackers and defenders alive at time t. The expected decrease in b(t) caused by a defender killing an attacker is represented by the term $\alpha_D b(t)d(t)$. A similar argument is used to write the second expression of (B.1).

Before presenting the solution for (B.1), we carry out a similar analysis as for the BCD scenario presented in Chapter IV. From (B.1), we get:

$$\alpha_B \frac{db(t)}{dt} = \alpha_D \frac{dd(t)}{dt} \qquad (B.2)$$

Integrating both sides of (B.2), we obtain:

$$\alpha_B[b(t) - b(0)] = \alpha_D[d(t) - d(0)]$$

from the initial conditions, we get:

$$\alpha_B[b(t) - B(0)] = \alpha_D[d(t) - D(0)] \quad \forall t \qquad (B.3)$$

Criteria for Defenders to Win

A rough criterion for defenders to win can now be obtained by assuming that in order to conclude that defenders won, we must have $b(\infty) = 0$ and $d(\infty) > 0$. Under this assumption, and as $t \to \infty$, (B.3) becomes:

$$\alpha_D d(\infty) = \alpha_D D(0) - \alpha_B B(0) \tag{B.4}$$

Since α_B, α_D, and $d(\infty)$ are all positive values, then:

$$\alpha_D D(0) - \alpha_B B(0) > 0$$

Therefore, the condition for the defenders to win is to satisfy the following condition:

$$\alpha_D D(0) > \alpha_B B(0) \tag{B.5}$$

Notice that equation (B.5) is the same as equation (4.9) which gives the condition for the defenders to win in air-to-air combat under the BCD scenario. So the deterministic model for the LLL model, which is usually used to represent land combat, and the BCD model resulted in the same condition for the defenders to win (eventually). This is due to the fact that the condition given by equation (4.9) (hence, (B.5)) is independent of the engagement rate θ, and also because of the fact that the product, $b(t)d(t)$, in (4.1) and (B.1) can be replaced by any function with the same result.

The solution for equation (B.1) is given by Taylor (1983):

$$b(t) = \begin{cases} B\left[\dfrac{\alpha_B B - \alpha_D D}{\alpha_B B - \alpha_D D \exp[-(\alpha_B B - \alpha_D D)t]}\right] & \text{for } \alpha_D D \neq \alpha_B B \\ \dfrac{B}{1 + \alpha_B Bt} & \text{for } \alpha_D D = \alpha_B B \end{cases} \tag{B.6}$$

and

$$d(t) = \begin{cases} D(0)\left[\dfrac{\alpha_D D - \alpha_B B}{\alpha_D D - \alpha_B B \exp[-(\alpha_D D - \alpha_B B)t]}\right] & \text{for } \alpha_D D \neq \alpha_B B \\ \dfrac{D}{1 + \alpha_D Dt} & \text{for } \alpha_D D = \alpha_B B \end{cases}$$

These equations permit easy numerical computation of $b(t)$, and $d(t)$, i.e., the representations for the number of combatants of both forces at various points in time during the combat period.

C. DIFFUSION MODEL FOR THE LLL SCENARIO

Following the arguments used in Chapter IV to derive the diffusion approximation for the BCD scenario, we can formulate the combat process under the LLL scenario as a diffusion process.

Define the following:

- $\tilde{B}(t)$ to be a stochastic diffusion representation for the number of attackers alive at time t.
- $\tilde{D}(t)$ to be a diffusion representation for the number of defenders alive at time t.

- N to be the total number of combatants of both forces available for combat initially (i.e., $N \equiv B(0) + D(0)$)

We can characterize the land combat process approximately when $N \to \infty$ (hence, $B(0) + D(0) \to \infty$), by treating $\{\tilde{B}(t), \tilde{D}(t);\ t \geq 0\}$ as a bivariate diffusion process. We can model the state vector $\{\tilde{B}(t), \tilde{D}(t);\ t \geq 0\}$ by writing it in the form of Ito stochastic differential equations as follows:

$$
\begin{aligned}
d\tilde{B}(t) &= -\alpha_D \tilde{B}(t)\tilde{D}(t)dt - \sqrt{\alpha_D \tilde{B}(t)\tilde{D}(t)}\ dW(t) \\
d\tilde{D}(t) &= -\alpha_B \tilde{B}(t)\tilde{D}(t)dt + \sqrt{\alpha_B \tilde{B}(t)\tilde{D}(t)}\ dW(t)
\end{aligned}
\tag{B.7}
$$

where $\{W(t);\ t \geq 0\}$ is a standard Wiener process.

Consider the following transformation:

$$
\begin{aligned}
\tilde{B}(t) &= N\tilde{b}(t) + \sqrt{N}\ X_1(t) \\
\tilde{D}(t) &= N\tilde{d}(t) + \sqrt{N}\ X_2(t)
\end{aligned}
\tag{B.8}
$$

Differentiating (B.8), we obtain:

$$
\begin{aligned}
d\tilde{B}(t) &= N\ d\tilde{b}(t) + \sqrt{N}\ dX_1(t) \\
d\tilde{D}(t) &= N\ d\tilde{d}(t) + \sqrt{N}\ dX_2(t)
\end{aligned}
\tag{B.9}
$$

where: $\tilde{b}(t)$ and $\tilde{d}(t)$ are deterministic functions of time, being approximations to the process means $\{X_i(t);\ t \geq 0\}$,

$i = 1,2$, are stochastic elements, all of which need to be determined.

Suppose that $X_1(0) = X_2(0) = 0$. Then we obtain,

$$\tilde{b}(0) = \frac{B(0)}{N} \tag{B.10}$$

and

$$\tilde{d}(0) = \frac{D(0)}{N}$$

Let $\tilde{\alpha}_B = \alpha_B N$ and $\tilde{\alpha}_D = \alpha_D N$ to be constants as $N \to \infty$; i.e., assume that α_B and α_D to be small enough such that $\alpha_B N$ and $\alpha_D N \to$ constant as $N \to \infty$. This means that on the average a defender or an attacker takes a relatively long time to achieve a kill.

Now, substituting (B.8) and (B.9) into (B.7), and isolating terms of order N and $\sqrt{N}$, and letting $N \to \infty$, we obtain the following sets of equations.

1. Deterministic Equations

The terms of order N yield:

$$\begin{aligned} \frac{d\tilde{b}(t)}{dt} &= -\tilde{\alpha}_D \tilde{b}(t)\tilde{d}(t) \\ \frac{d\tilde{d}(t)}{dt} &= -\tilde{\alpha}_B \tilde{b}(t)\tilde{d}(t) \end{aligned} \tag{B.11}$$

with initial conditions given by (B.10).

Equations (B.11) are equivalent to equation (B.1), which shows that the diffusion approximation for the LLL scenario results in the same approximations for the process means as the deterministic model.

2. <u>Stochastic Equations</u>

The terms of order $\sqrt{N}$ yield:

$$
\begin{aligned}
dX_1(t) &= -\{\tilde{\alpha}_D \tilde{d}(t) X_1(t) + \tilde{\alpha}_D \tilde{b}(t) X_2(t)\ dt - \sqrt{\tilde{\alpha}_D \bar{b}(t) \bar{d}(t)}\ dW(t) \\
dX_2(t) &= -\{\tilde{\alpha}_B \tilde{d}(t) X_1(t) + \tilde{\alpha}_B \tilde{b}(t) X_2(t)\ dt + \sqrt{\tilde{\alpha}_B \bar{b}(t) \bar{d}(t)}\ dW(t)
\end{aligned}
\qquad \text{(B.12)}
$$

Writing the above in a matrix form, we get:

$$d\vec{X}(t) = A(t)\vec{X}(t)dt + \vec{V}(t)d\vec{W}(t)$$

where:

$$\vec{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

and

$$\vec{V}(t) = \begin{bmatrix} -\sqrt{\tilde{\alpha}_D \bar{b}(t) \bar{d}(t)} \\ \sqrt{\tilde{\alpha}_B \bar{b}(t) \bar{d}(t)} \end{bmatrix}$$

and

$$A(t) = \begin{bmatrix} -\tilde{\alpha}_D \tilde{d}(t) & -\tilde{\alpha}_D \tilde{b}(t) \\ -\tilde{\alpha}_B \tilde{d}(t) & -\tilde{\alpha}_B \tilde{b}(t) \end{bmatrix}$$

Since $\vec{X}(0) = \vec{0}$, $\tilde{b}(0)$ and $\tilde{d}(0)$ are given by (B.10). Then by appealing to the central limit theorem, for all $t > 0$, $\{X_1(t), X_2(t);\ t \geq 0\}$ has a bivariate normal distribution with mean $\vec{0}$ and covariance matrix $\Sigma(t)$ which satisfies the following differential equation (Arnold, 1974):

$$\frac{d}{dt}\Sigma(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + \vec{V}(t)\vec{V}^T(t) \tag{B.13}$$

Recall that

$$\begin{bmatrix} \tilde{B}(t) \\ \tilde{D}(t) \end{bmatrix} = N \begin{bmatrix} \tilde{b}(t) \\ \tilde{d}(t) \end{bmatrix} + \sqrt{N} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

Therefore, we have obtained the following:

<u>RESULT (B.1)</u>:

$\{\tilde{B}(t), \tilde{D}(t)\}$ is approximately bivariate normal (Gaussian), as $N \to \infty$ (hence, $B(0)$ and $D(0) \to \infty$ simultaneously and in a fixed proportion).

$$\{\tilde{B}(t),\tilde{D}(t)\} \approx \text{Normal}(N(b(t),d(t)),N\Sigma(t))$$

By the required substitution and multiplication, we get:

$$\vec{V}(t)\vec{V}^T(t) = \begin{bmatrix} \tilde{\alpha}_D\tilde{b}(t)\tilde{d}(t) & -\tilde{b}(t)\tilde{d}(t)\sqrt{\tilde{\alpha}_B\tilde{\alpha}_D} \\ -\tilde{b}(t)\tilde{d}(t)\sqrt{\tilde{\alpha}_B\tilde{\alpha}_D} & \tilde{\alpha}_B\tilde{b}(t)\tilde{d}(t) \end{bmatrix}$$

Writing:

$$\Sigma(t) = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) \end{bmatrix}$$

Equation (B.13) becomes:

$$\frac{d}{dt}\vec{S}(t) = G(t)\vec{S}(t) + H(t)$$

with initial conditions:

$$\vec{S}(\vec{0}) = \vec{0}$$

where:

$$\vec{S}(t) = (\sigma_{11}(t) \quad \sigma_{12}(t) \quad \sigma_{22}(t))$$

$$G(t) = \begin{bmatrix} -2\tilde{\alpha}_D\tilde{d}(t) & -2\tilde{\alpha}_D\tilde{b}(t) & 0 \\ -\tilde{\alpha}_B\tilde{d}(t) & -(\tilde{\alpha}_B\tilde{b}(t)+\tilde{\alpha}_D\tilde{d}(t) & -\tilde{\alpha}_D\tilde{b}(t) \\ 0 & -2\tilde{\alpha}_B\tilde{d}(t) & -2\tilde{\alpha}_B\tilde{b}(t) \end{bmatrix}$$

and

$$H(t) = \begin{bmatrix} \tilde{\alpha}_D\tilde{b}(t)\tilde{d}(t) \\ -\tilde{b}(t)\tilde{d}(t)\sqrt{\tilde{\alpha}_B\tilde{\alpha}_D} \\ \tilde{\alpha}_B\tilde{b}(t)\tilde{d}(t) \end{bmatrix}$$

D. LLL MARKOVIAN COMBAT MODEL

Suppose an attacker takes an exponential amount of time with mean α_B^{-1} to kill a defender; and that a defender takes an exponential amount of time with mean α_D^{-1} to kill an attacker. Then the combat process can be represented by the bivariate-continuous-time discrete-state Markov process $\{B(t),D(t);\ t \geq 0\}$ operating on the state space $S = \{(i,j): 0 < i+j;\ i = 0,1,\ldots,B;\ j = 0,1,\ldots,D\}$.

The transition probabilities of the process $\{B(t),D(t);\ t \geq 0\}$ of order dt are as follows:

t		t+dt	Probability	
(i,j)	$\rightarrow$	$(i,j-1)$	$f_B(i,j)dt + o(dt)$	
	$\rightarrow$	$(i-1,j)$	$f_D(i,j)dt + o(dt)$	(B.14)
	$\rightarrow$	(i,j)	$1 - f(i,j)dt + o(dt)$	

where:

$$f_B(i,j) = \alpha_B ij$$

$$f_D(i,j) = \alpha_D ij$$

$$f(i,j) = f_B(i,j) + f_D(i,j)$$

Let

$$P_{mn,ij}(t) = P\{B(t) = i, D(t) = j | B(0) = m, D(0) = n\}$$

For simplicity of notation we will suppress the initial condition, and write:

$$P_{i,j}(t) = P_{mn;ij}(t)$$

Apparently $\{B(t), D(t);\ t \geq 0\}$ as defined by (B.14) is a finite state bivariate death process.

The forward Chapman-Kolmogorov equations are:

$$P'_{B,D}(t) = -f(B,D)P_{B,D}(t)$$

$$P'_{B,j}(t) = -f(B,j)P_{B,j}(t) + f_B(b,j+1)P_{B,j+1}(t) \quad 1 \leq j \leq D-1$$

$$P'_{i,D}(t) = -f(i,D)P_{i,D}(t) + f_D(i+1,D)P_{i+1,D}(t) \quad 1 \leq i \leq B-1 \tag{B.15}$$

$$P'_{i,j}(t) = -f(i,j)P_{i,j}(t) + f_B(i,j+1)P_{i,j+1}(t) + f_D(i+1,j)P_{i+1,j}(t) \qquad 1 \le i \le B-1, \quad 1 \le j \le D-1$$

$$P'_{0,j}(t) = f_D(1,j)P_{1,j}(t) \qquad 1 \le j \le D$$

$$P'_{i,0}(t) = f_B(i,1)P_{i,1}(t) \qquad 1 \le i \le B$$

(B.15)
(Cont'd)

with initial condition:

$$P_{i,j}(0) = \begin{cases} 1 & \text{if } (i,j) = (B,D) \\ 0 & \text{otherwise} \end{cases}$$

Let $\hat{P}_{i,j}(s)$ be the Laplace transform for $P_{i,j}(t)$. Then:

$$\hat{P}_{i,j}(s) = \int_0^\infty e^{-st} P_{i,j}(t)dt \qquad s > 0$$

We now take the Laplace transforms for (B.15) by following the procedure done for the BCD scenario; and using the initial condition we get:

$$\hat{P}_{B,D}(s) = \frac{1}{f(B,D)+s}$$

$$\hat{P}_{B,j}(s) = \frac{1}{f(B,j)+s}\{f_B(B,j+1)\hat{P}_{B,j+1}(s)\} \qquad 1 \le j \le D-1$$

$$\hat{P}_{i,D}(s) = \frac{1}{f(i,D)+s}\{f_D(i+1,D)\hat{P}_{i+1,D}(s)\} \qquad 1 \le i \le B-1$$

$$\hat{P}_{i,j}(s) = \frac{1}{f(i,j)+s}\{f_B(i,j+1)\hat{P}_{i,j+1}(s) + f_D(i+1,j)\hat{P}_{i+1,j}(s)\} \qquad \text{(B.16)}$$

$$1 \le j \le B-1$$

$$1 \le j \le D-1$$

$$\hat{P}_{0,j}(s) = \frac{1}{s}\, f_D(1,j)\hat{P}_{1,j}(s) \qquad 1 \le j \le D$$

$$\hat{P}_{i,0}(s) = \frac{1}{s}\, f_B(i,1)\hat{P}_{i,1}(s) \qquad 1 \le i \le B$$

Figure (B.1) shows a graph representation for the Markovian combat process. Equations (B.16) are in a form that is suitable for a recursive solution.

$\hat{P}_{B,D}(s)$ is given in (B.16). Now we can find $\hat{P}_{B,D-1}(s)$ and $\hat{P}_{B-1,D}(s)$ since they are functions of $\hat{P}_{B,D}(s)$ only. The $\hat{P}_{B,D-1}(s)$ and $\hat{P}_{B-1,D}(s)$ are the only transforms needed to find $\hat{P}_{B,D-2}(s)$, $\hat{P}_{B-1,D-1}(s)$, and $\hat{P}_{B-2,D}(s)$, as given by (B.16). Continuing this way, we find, as in the BCD Markovian model, that by knowing the Laplace transforms of the states in the n^{th} depth of the graph, the Laplace transforms for the $(n+1)^{st}$

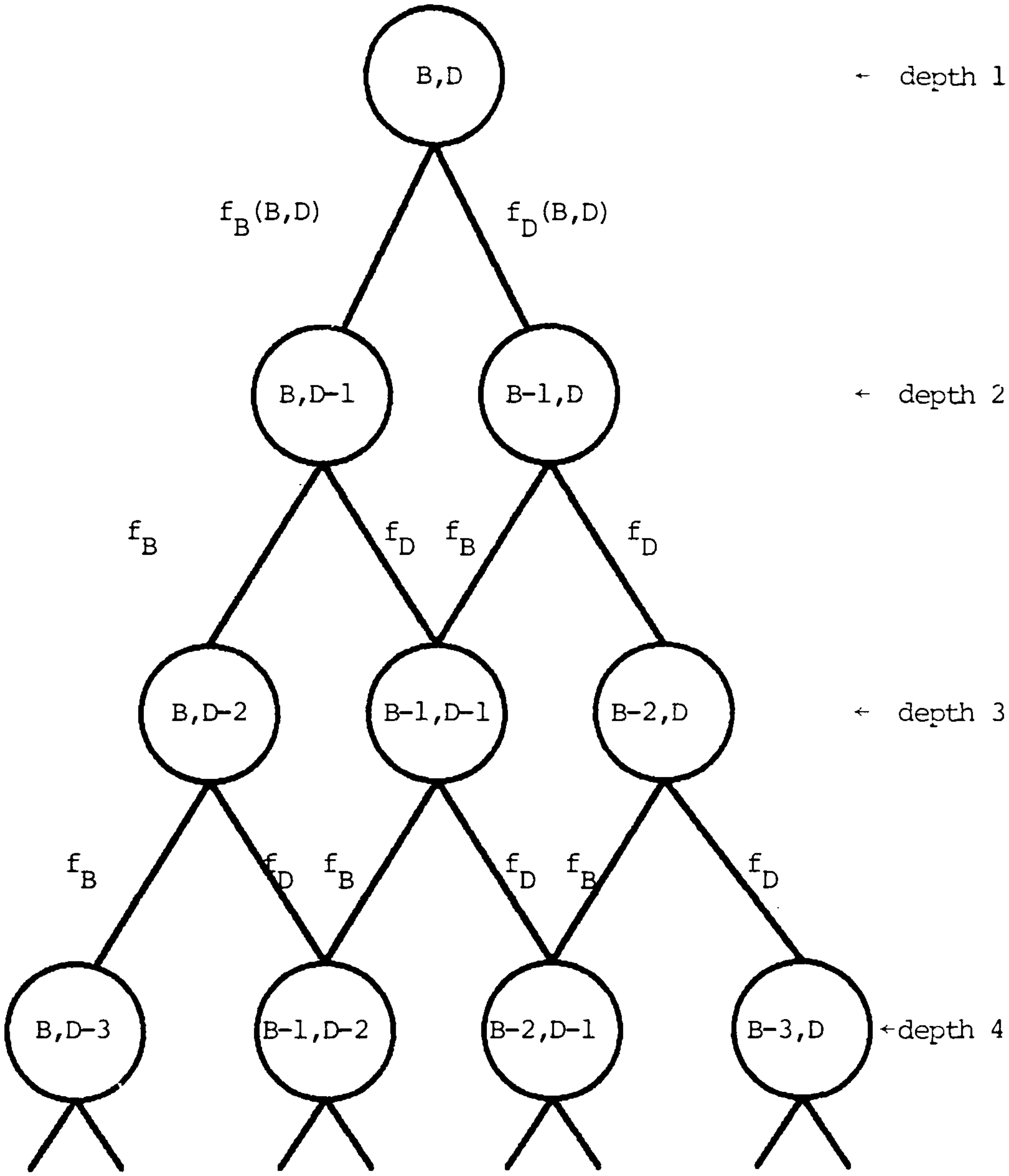

Figure B.1. A Graph Representation of the LLL Combat Process

depth can be calculated for any value of s. Therefore, an algorithm similar to the algorithm used to solve (4.49) can now be constructed to solve (B.16):

1. Construct the graph of the process as follows: For the n^{th} depth, find the possible states and calculate its width, i.e., how many states are in the n^{th} depth. Continue until the width of the depth is zero.

2. For the n^{th} depth, if the width = 0, stop. Else, pick a state (say (i,j)). Then:

   a. Calculate the mean of the corresponding sojourn time of the state (i.e., $(f(i,j))^{-1}$).

   b. Examine all states in the $(n-1)^{st}$ depth to determine states that are accessible to state (i,j).

   c. Calculate the corresponding transition rates from the states determined in b. to state (i,j).

   d. Calculate the Laplace transform for (i,j). Repeat for all states in the n^{th} depth.

The solutions for (B.16), i.e., the Laplace transforms, can be used to evaluate quantities of interest, such as:

$$\pi_D(i) \quad = \quad P\{i \text{ defenders eventually survive combat}\}$$

Following the same arguments presented for the BCD model, it can be shown that:

$$\pi_D(i) \quad = \quad f_D(1,j)\hat{P}_{1,j}(0) \tag{B.17}$$

and

$$E[B(\infty)\,|B(0) = B, D(0) = D] \quad = \quad \sum_{j=1}^{D} j f_D(1,j)\hat{P}_{1,j}(0) \tag{B.18}$$

E. SIMULATION OF LLL MARKOVIAN SCENARIO

Following the same arguments used for simulating the BCD scenario, we can simulate the LLL Markovian combat process.

The Monte-Carlo simulation was constructed based on the following:

a. The time spent by the process at state (i,j) is exponentially distributed with rate f(i,j).

b. The transition probabilities for the embedded Markov chain, are as follows:

$$\begin{array}{llll} \underline{\text{Present}} & & \underline{\text{Next}} & \underline{\text{Probability}} \\ (i,j) & \rightarrow & (i,j-1) & \dfrac{f_B(i,j)}{f(i,j)} \\ & \rightarrow & (i-1,j) & \dfrac{f_D(i,j)}{f(i,j)} \end{array} \tag{B.19}$$

Generate U_1 and U_2 as in the BCD simulation. A realization of $X_{i,j}$, time spent in state (i,j) can be obtained by:

$$X_{i,j} = \frac{-\ln U_1}{f(i,j)} \tag{B.20}$$

After spending $X_{i,j}$ units of time in state (i,j), the process will transit to:

$$\begin{array}{lll} (i,j-1) & \text{if} & 0 < U_2 \le \dfrac{f_B(i,j)}{f(i,j)} \\ (i-1,j) & & \text{otherwise} \end{array} \tag{B.21}$$

Let $\{B_m(n\Delta t), D_m(n\Delta t);\ M = 1,2,\ldots,M\}$ be a (the m^{th}) realization of the process $\{B(t), D(t)\}$ at time $n\Delta t$, and during the m^{th} replication of the simulation model.

Defining BB, BS, DD, and DS as in the BCD simulation, we can calculate the statistics by applying equations (4.85) and (4.86).

The following basic data were used as an example for model illustration.

A threat of $B(0) = 12$ attackers engaging exactly $D(0)$ defenders. Each defender and attacker is assumed to take an exponential amount of time with mean $(\alpha_D)^{-1} = (.008)^{-1}$ minutes, and $(\alpha_B)^{-1} = (.004)^{-1}$ minutes respectively, to kill an opponent.

Suppose we are interested in evaluating $P\{B(t) > 3 | B(0) = 12, D(0) = j\}$ for $j = 1,2,\ldots,20$. FORTRAN programs were written for:

1. Computing the stochastic mean, variance and probability distribution for B(t) using the forward Chapman-Kolmogorov equations.
2. Computing the approximate mean and variance using the diffusion approximation to compute the required probability.
3. Simulating the Markov process to compute the stochastic mean, variance, and required probabilities.

For illustration purposes the following values for D(0) were considered:

$$D(0) = 3,\ 5,\ 10,\ 12,\ 16 \text{ and } 20$$

Figure (B.2) shows a plot of the stochastic mean resulting from the simulation model and the deterministic mean resulting from the diffusion model. It is clear from the plot that there exists a bias between the stochastic mean and the deterministic mean. The bias $\rightarrow 0$ as $D(0)$ increases. It is also clear that the deterministic mean underestimates the stochastic mean.

Figure (B.3) shows a plot for the variance function resulting from simulation compared to the variance function resulting from diffusion approximation, for the various values of $D(0)$. We find that the variance of the diffusion tends to approach that provided by simulation as $D(0)$ increases. Figures (B.2) and (B.3) show that the diffusion approximation model results in good approximation as $D(0)$ increases. The example shows that for a relatively small value of $D(0)$, e.g., $D(0) = 16$ and $D(0) = 20$, the diffusion approximation started to result in a good approximation.

Since there exists bias in both of the mean and variance approximations, we therefore expect to have some difference between the actual probabilities for the Markov process and the probabilities resulting from applying the normal approximation. To compare the different values for $P\{B(t) > 3 | B(0), D(0)\}$ a value of $t = 20$ minues was selected.

Figure (B.4) shows a comparison of the actual values for $P\{B(t) > 3 | B(0), D(0)\}$, as a result of solving the forward Chapman-Kolmogorov equations numerically, to the normal approximation values with continuity correction using the

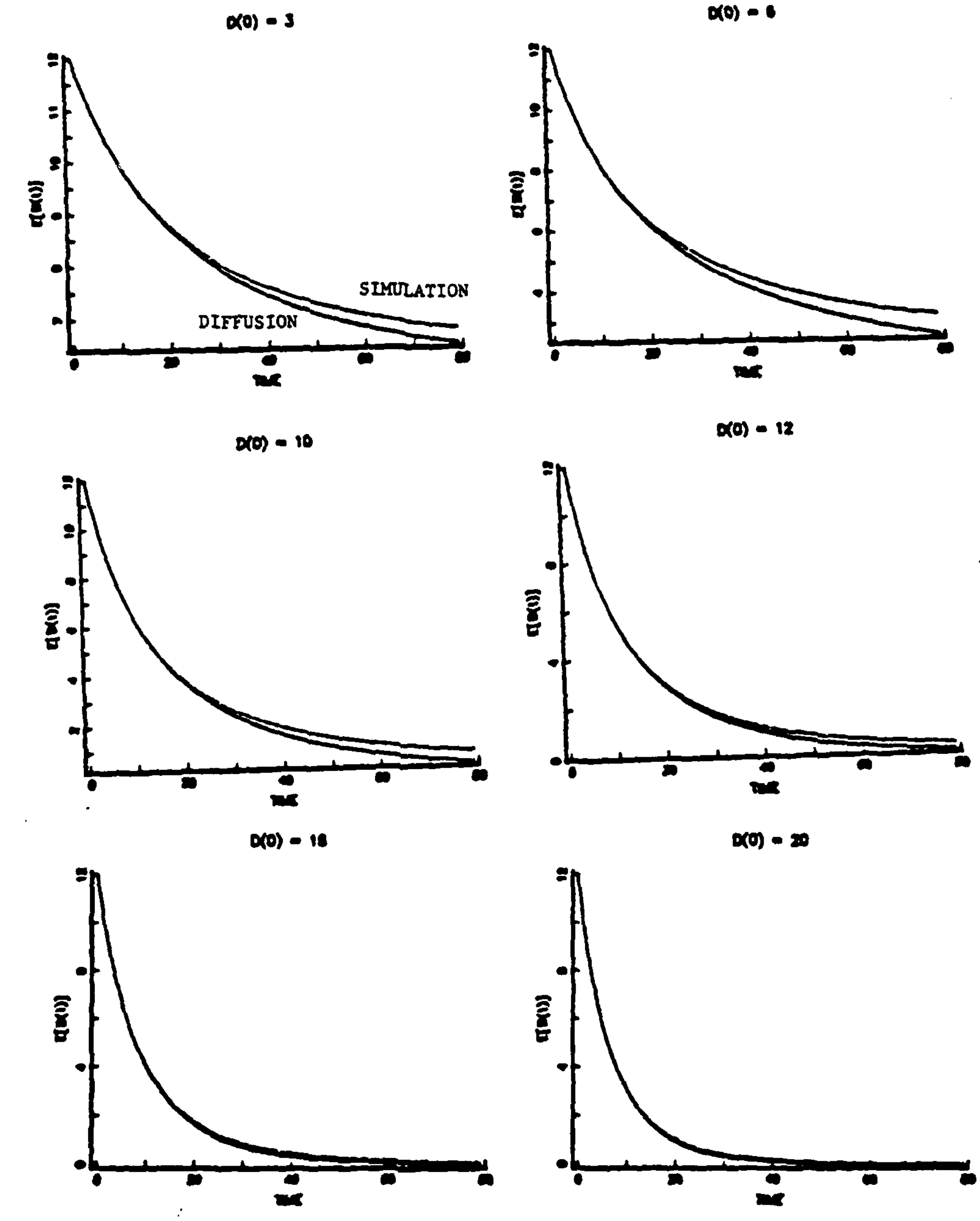

Figure B.2. Deterministic Mean Versus the Stochastic Mean of B(t): LLL Model

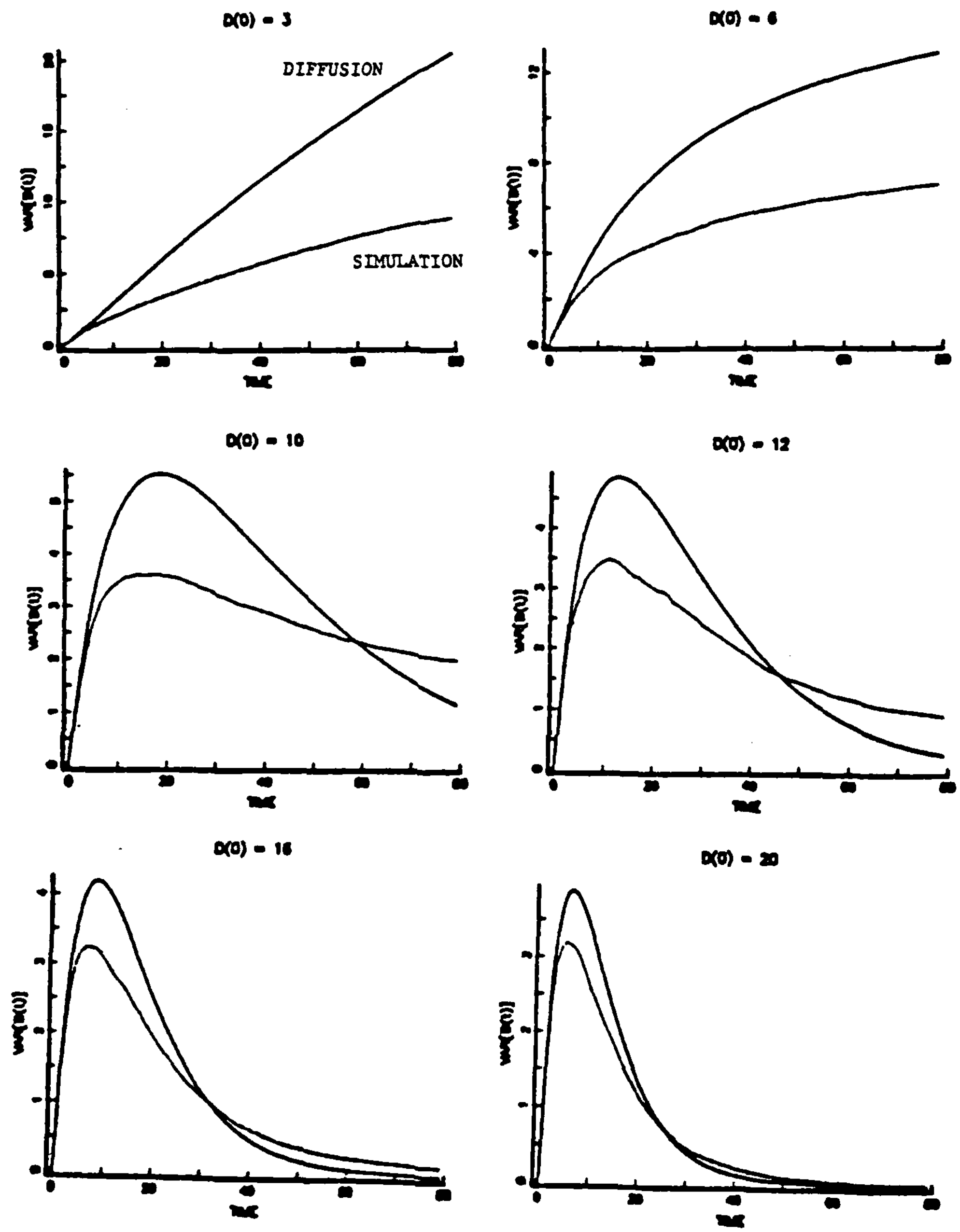

Figure B.3. Plots of the Variance Approximation Resulting from Diffusion Versus Variance from Simulation of B(t): LLL Scenario

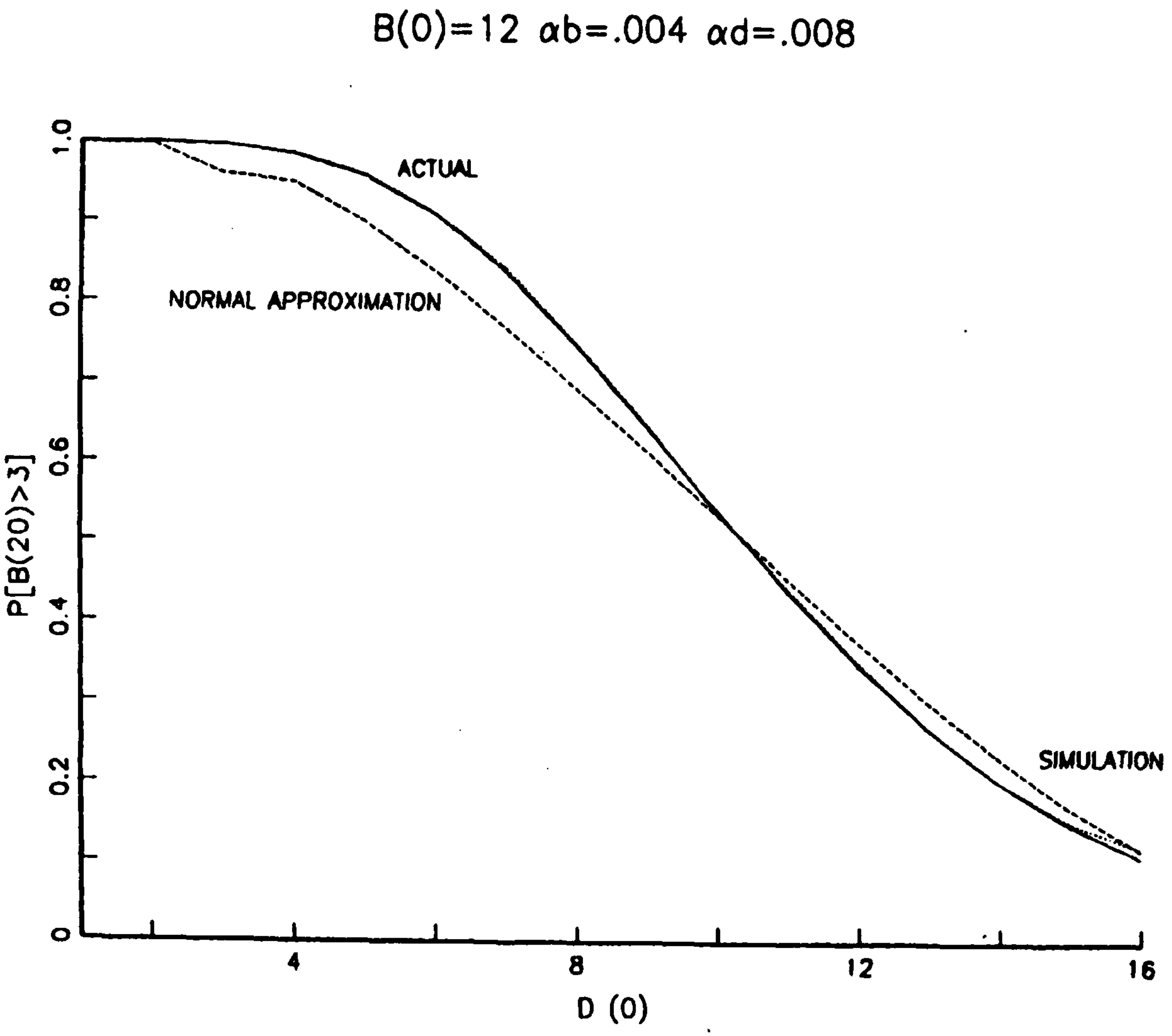

Figure B.4. Comparison of Actual Value, to Normal Approximation with Continuity Correction and to the Simulation Results, of $P\{B(20) > 3 | B(0) = 12, D(0) = j\}$ for $j = 1,2,\ldots,16$

diffusion approximation, and the probability resulting from the simulation model. We find that the simulation output represents the actual probability obtained by solving the Chapman-Kolmogorov equations. The diffusion tends to give a better approximation as $D(0) \to \infty$.

Figure (4.5) shows plots for the various differences between the probabilities which supports the above observation.

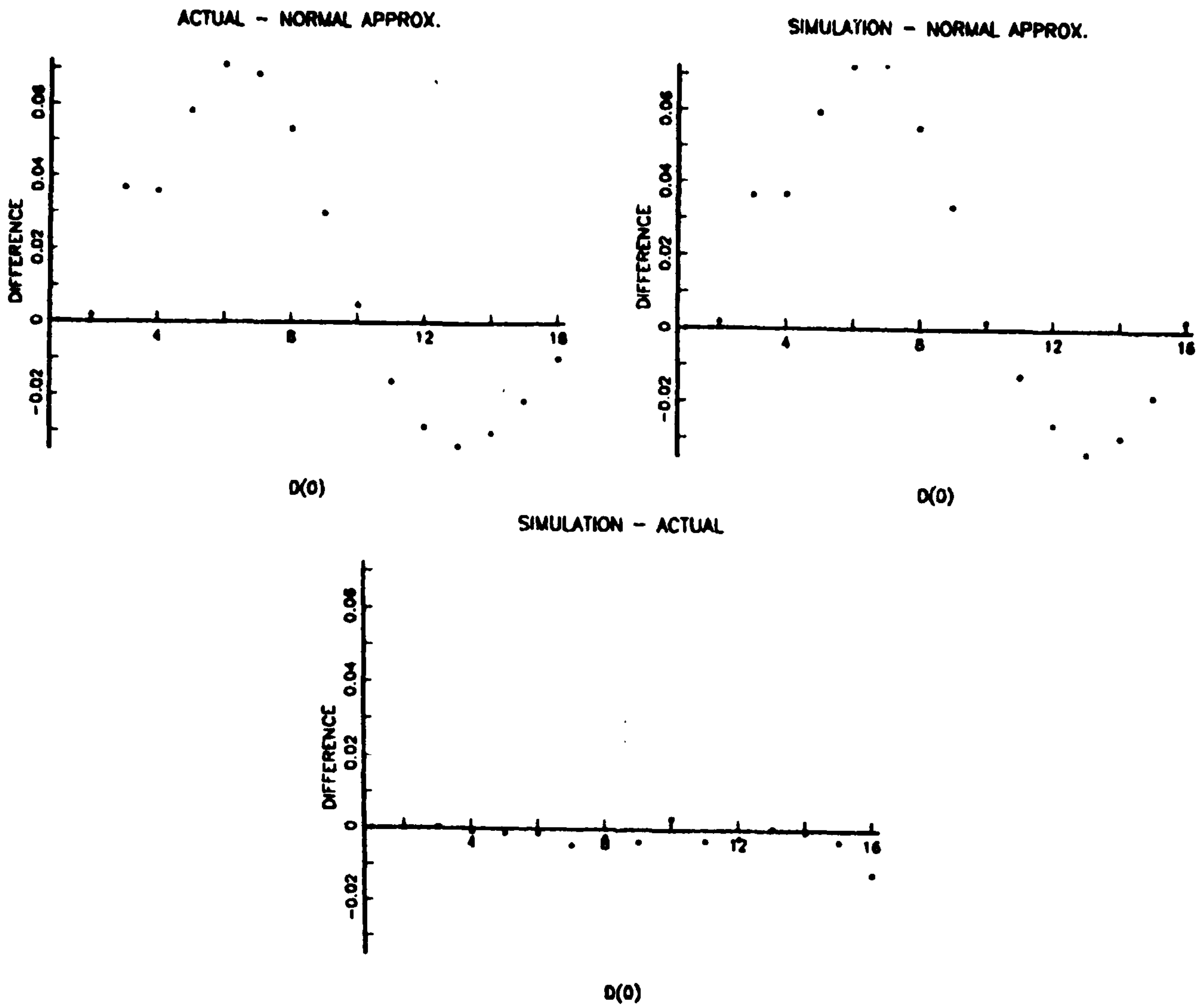

Figure B.5. Differences of the Various Values of the Probabilities Shown in Figure 4.9

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